A UNIFIED APPROACH TO RANK TESTS FOR MIXED MODELS

Michael G. Akritas and Edgar Brunner *

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*Michael G. Akritas is Professor, Department of Statistics, The Pennsylvania State University, State College PA 16802, U.S.A. Edgar Brunner is Professor, Department of Medical Statistics, University of Göttingen, Humboldt Allee 32, D-37073 Göttingen, Germany. Part of this paper was prepared during the second author's visit to the Department of Statistics, Pennsylvania State University, State College. He likes to thank the Department of Statistics for its hospitality and support. The collaboration was supported in part by NATO grant CRG 950178. The authors are grateful to the referees and to the associate editor for their helpful comments which improved the first draft of the paper considerably.

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Abstract

The nonparametric version of the classical mixed model is considered and the common hypotheses of (parametric) main effects and interactions are re-formulated in a nonparametric setup. To test these nonparametric hypotheses, the asymptotic distributions of quadratic forms of rank statistics are derived in a general framework which enables the derivation of the statistics for the nonparametric hypotheses of the fixed treatment effects and interactions in an arbitrary mixed model. The procedures given here are not restricted to semiparametric models or models with additive effects. Moreover, they are robust to outliers since only the ranks of the observations are needed. They are also applicable to pure ordinal data and since no continuity of the distribution functions is assumed, they can also be applied to data with ties. Some approximations for small sample sizes are suggested and analyzed in a simulation study. The application of the statistics and the interpretation of the results is demonstrated in several worked-out examples where some data sets given in the literature are re-analyzed.

Abbreviated title:

Rank Tests for Mixed Models

KEY WORDS: Rank transform; Factorial designs; Nonparametric hypotheses; Mixed model; Rank tests; Ordinal data; Ties

1 Introduction

Background and historical remarks In a mixed model, randomly chosen subjects are observed repeatedly under the same or under different treatments. Such designs occur in many biological experiments, in the behavioral sciences and ecological, medical or psychological studies. The subjects are the levels of the random factor(s) and the subject effects are regarded as unobservable random variables.

The two main assumptions underlying the classical analysis of variance (ANOVA) models are the linear model and the normal distribution of the error term. Our aim is to generalize the classical models of ANOVA in such a way that not only the assumption of normality of the error terms will be relaxed but also the structure of the designs will be introduced in a broader framework. Based on the ideas of Akritas & Arnold (1994) and Akritas, Arnold & Brunner (1994), we will formulate nonparametric hypotheses in the various designs and derive nonparametric tests for these hypotheses. In order to identify the testing problem underlying the different rank procedures, the relations between the hypotheses in the nonparametric model and in the common linear model will be investigated.

Nonparametric hypotheses and tests for the mixed model have already been considered by Sen (1967), Koch & Sen (1968) and by Koch (1969, 1970). Mainly joint hypotheses in the linear model are considered, i.e. main effects and certain interactions are tested together. Moreover, some of the statistics given there are not pure rank statistics rather than aligned rank statistics and therefore they are restricted to linear models. For the simple mixed model with two treatments for paired observations, rank tests using overall ranks on the original observations have been considered by Hollander, Pledger & Lin (1974) and Govindarajulu (1975). In the former paper, the robustness of the Wilcoxon-Mann-Whitney statistic with respect to deviation from independence is studied. In the latter paper, a rank statistic is derived and it is indicated how the unknown variance may be estimated. Lam & Longnecker (1983) derived an estimator for the unknown variance based on Spearman's rank correlation. Conover & Iman (1981) suggested to perform a paired t-test on the ranks; that the paired t-test on the ranks is a valid procedure was established in the more general context of censored data by Akritas (1992). Brunner & Neumann (1982, 1984, 1986a, b) derived the asymptotic distribution of rank statistics in two-factor mixed models with an equal number of replications and applied the results to different mixed models. The asymptotic variances and covariance matrices of the statistics were estimated using ranks over all observations and ranks within the treatments. Rank tests for the mixed model with $m = 1$ replication were considered by Kepner & Robinson (1988). Thompson (1990) generalized the results of Brunner & Neumann (1982) and derived the asymptotic distribution of a linear rank statistic for independent vectors of an equal fixed length and she applied the result to various balanced mixed models (Thompson & Ammann (1989, 1990), Thompson (1991)) and derived statistics for joint hypotheses where main effects and interactions are tested together. The assumption of an equal fixed length of the vectors was used in the proofs of the asymptotic normality of the statistics in the papers by Brunner & Neumann (1982) and Thompson (1990). Akritas (1993) derived a rank test in a special unbalanced mixed model. A general result for the asymptotic normality of a linear rank statistic for vectors of unequal length was derived by Brunner & Denker (1994) and the results were applied to various unbalanced mixed models.

In the above papers, all theoretical results as well as the derivation of the rank statistics were based on the assumption of continuous distribution functions and ties were excluded although some theoretical results (e.g. Conover (1973) and Behnen (1976)) regarding ties were available in the literature for linear rank statistics of independent observations. Koch & Sen (1968) and Koch (1970) recommend to breake ties by using the 'mid-rank-method'. This method was used by Boos & Brownie (1992) to derive estimators of the asymptotic variance of a rank statistic using the U-statistic representation. Munzel (1994) generalized the results of Brunner & Denker (1994) to the case of ties and the same method is used by Brunner, Puri & Sun (1995) to derive the asymptotic normality of linear rank statistics and consistent estimators of the asymptotic variance for fixed and mixed $2 \times b$ designs.

Based on theses ideas, we give a unified approach to mixed models for nonparametric hypotheses. The statistics considered in the literature quoted above follow as special cases from our general approach (except that we use only Wilcoxon scores, for simplicity). Moreover, main effects and interactions, defined through the nonparametric hypotheses, can be tested separately. We will consider only (pure) rank statistics because they are invariant under strictly monotone transformations of the data and they are robust to outliers. We do not assume the continuity of the distribution functions so that also data with arbitrary (non-trivial) ties may be evaluated with the procedures given in this paper. In particular, these procedures are applicable to ordinal data such as scores in psychological tests, and quality scales in order to describe the degree of the damage of plants or trees in ecological or environmental studies. We shall not consider procedures which need sums or differences of the original data and which are therefore restricted to the linear model. Procedures for ordered alternatives, many-one and multiple comparisons follow from the general approach. However, they shall not be the subject of this paper and are not considered here. Regarding rank procedures for heteroskedastic mixed models, we refer to Brunner & Denker (1994), Brunner, Puri & Sun (1995) and Brunner & Puri (1996).

Applications. The general results to be derived in Section 2, will be applied to four different designs.

(1) **Two-factor mixed models:**

- (a) random factor and fixed factor crossed,
- (b) random factor nested under the fixed factor.

(2) **Three-factor mixed models with two factors fixed:**

- (a) repeated measurements on one fixed factor, the random factor is crossed with the fixed factor B and is nested under the fixed factor A,
- (b) repeated measurements on both crossed fixed factors.

We distinguish two models:

(I) The repeated measurements model where in the case of 'no treatment effect' the common distribution function of the observed random variables on subject i is invariant under the numbering of the treatment levels. This means that the random variables within one subject are interchangeable and their joint distribution function is compound symmetric. Note that in the linear mixed model with independent random effects, the compound symmetry of the joint distribution functions of the subjects is preserved under the alternative. In general models however (to be stated in the following sections), it seems to be unrealistic to assume that compound symmetry is preserved if treatment effects are present.

(II) The multivariate model allows arbitrary dependencies between the observed random variables within one subject. This is typically the case with longitudinal data or inhomogeneous materials. A multivariate model is also assumed if a treatment effect is present in a repeated measurements model. We do not consider special patterns of dependencies such as an autocorrelation structure, e.g.

In this paper, we will consider only the multivariate model for brevity.

General notation. The distribution function of a random variable X is defined by

$$F(x) \;=\; \frac{1}{2}\left[F^+(x) + F^-(x)\right] \tag{1}$$

where $F^+(x) = P(X \leq x)$ is the right continuous version and $F^-(x) = P(X < x)$ is the left continuous version of the distribution function. This definition of the distribution function includes the case of ties. We exclude only the trivial case when $F(x)$ is a one-point distribution function. We write $X \sim F$ to indicate that F is the distribution function of X. The asymptotic equivalence ($N \to \infty$) of two sequences of random variables X_N and Y_N is denoted by $X_N \doteq Y_N$.

Let $X_s \sim F$, $s = 1, \ldots, m$, let $\widehat{F}^+$ denote the usual right-continuous version of the empirical distribution function based on the X_s, and let $\widehat{F}^-$ be its left-continuous version. Then the empirical distribution function of F is denoted by

$$\widehat{F}(x) \;=\; \frac{1}{2}\left[\widehat{F}^+(x) + \widehat{F}^-(x)\right] = \frac{1}{m}\sum_{s=1}^{m} c(x - X_s) \tag{2}$$

where $c(u) = \frac{1}{2}\left[c^+(u) + c^-(u)\right]$ denotes the counting function and $c^+(u) = 0$ or 1 according as $u <$ or ≥ 0 and $c^-(u) = 0$ or 1 according as $u \leq$ or > 0. The rank of a random variable X_s among all m random variables $X_1, \ldots, X_m$ is defined as

$$R_s \;=\; \frac{1}{2} + m\widehat{F}(X_s) = \frac{1}{2} + \sum_{j=1}^{m} c(X_s - X_j). \tag{3}$$

Note that 1/2 has to be added to $m\widehat{F}(X_s)$ in order to get the position numbers of the ordered observations in case of no ties since $c(0) = 1/2$ as defined above. Note also that R_s is the midrank in case of ties.

For the formulation of the hypotheses, we will use the contrast matrices

$$\mathbf{C}_1 = (\mathbf{1}_{c-1} | -\mathbf{I}_{c-1}) \quad \text{and} \quad \mathbf{C}_2 = \mathbf{P}_c = \mathbf{I}_c - c^{-1}\mathbf{J}_c \tag{4}$$

where $\mathbf{1}_c = (1, \ldots, 1)'$ denotes the c-dimensional vector of 1's, $\mathbf{I}_c = diag\{1, \ldots, 1\}$ denotes the c-dimensional unit matrix and $\mathbf{J}_c = \mathbf{1}_c\mathbf{1}_c'$ denotes the c-dimensional square matrix of 1's. Note that $rank(\mathbf{C}_1) = c - 1$ and therefore, $\mathbf{C}_1\mathbf{C}_1'$ is nonsingular. The matrix $\mathbf{C}_2$ is a c-dimensional projection matrix of rank $c - 1$. We will also use the contrast matrix

$$\mathbf{W} \;=\; \mathbf{V}^{-1}\left(\mathbf{I}_c - \mathbf{J}_c\mathbf{V}^{-1}/(\mathbf{1}_c'\mathbf{V}^{-1}\mathbf{1}_c)\right) \tag{5}$$

where the nonsingular covariance matrix $\mathbf{V}$ is a generalized inverse of $\mathbf{W}$, i.e. $\mathbf{WVW} = \mathbf{W}$ (see Lemma A.1 in the Appendix).

Factors (in the sense of experimental design) will be denoted by capital letters $A, B, C, \ldots$ and the levels of A are numbered by $i = 1, \ldots, a$, the levels of B are numbered by $j = 1, \ldots, b$, etc. If factor B is nested under factor A, this is denoted by $B(A)$.

For a convenient and technically simple description of hypotheses in two- and higher-way layouts, we will use the Kronecker-product $\mathbf{A} \otimes \mathbf{B}$ and the Kronecker-sum

$$\mathbf{A} \oplus \mathbf{B} \;=\; \left(\begin{array}{c|c} \mathbf{A} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{B} \end{array}\right)$$

of matrices. The Kronecker-product of matrices $\mathbf{A}_i$, $i = 1, \ldots, a$ is written as $\bigotimes_{i=1}^{a} \mathbf{A}_i$ and the Kronecker-sum is written as $\bigoplus_{i=1}^{a} \mathbf{A}_i$.

Model. The general mixed model can be formulated by independent random vectors

$$\mathbf{X}_{ik} \;=\; (\mathbf{X}_{i1k}', \ldots, \mathbf{X}_{ick}')' \;, \quad i = 1, \ldots, r \text{ and } k = 1, \ldots, n_i \tag{6}$$

where $\mathbf{X}_{ijk} = (X_{ijk1}, \ldots, X_{ijkm_{ijk}})'$, $j = 1, \ldots, c$ and $X_{ijks} \sim F_{ij}$, $k = 1, \ldots, n_i$ and $s = 1, \ldots, m_{ijk} \geq 1$. To avoid inconvenient notations, we consider only the case where $m_{ijk} \geq 1$, i.e. no missing values. We note however that models with missing (at random) values can also be treated by our general approach.

The $i-$th level of the row-factor, $i = 1, \ldots, r$, is applied to all c parts of each of the n_i subjects that are nested under the level i, and there are m_{ijk} replications at the (i, j, k) factor level and subject combination. The n_i subjects independent for each level i yield independent (vectors of) observations. If more than one factor is applied to the subjects, then the r levels may be regarded as a lexicographic ordering of all factor level combinations of the factors.

The column-factor with c levels is applied to all subjects. However, the level j of this factor is applied only to the j-th part of the subject (which is split into c homogeneous parts). This factor is crossed with the subjects. If more than one factor is applied to each subject, then the c levels may be regarded as a lexicographic ordering of all factor level combinations of the factors.

Examples

1. In the matched pairs design, we have n independent random vectors $\mathbf{X}_k = (X_{1k}, X_{2k})'$ where $X_{jk} \sim F_j$, $k = 1, \ldots, n$ and $j = 1, 2$. This design is derived from the general mixed model (6) by letting $r = 1$, $n_1 = n$, $c = 2$ and $m_{ijk} = 1$.

2. The two-factor hierarchical design is a special case of (6) if $r = a$, $c = 1$, $m_{ijk} = m_{ik}$ and $X_{ijks} = X_{iks} \sim F_i$, $i = 1, \ldots, a$, $k = 1, \ldots, n_i$ and $s = 1, \ldots, m_{ik}$.

3. For the one-factor block design with b treatment levels and with n blocks, we choose $r = 1$, $m_{ijk} = 1$, $X_{ijks} = X_{jk} \sim F_j$, $j = 1, \ldots, b$ and $k = 1, \ldots, n$.

4. The $a \times b$ split-plot design is a special case of (6) with $m_{ijk} = 1$, $X_{ijks} = X_{ijk} \sim F_{ij}$, $i = 1, \ldots, a$, $j = 1, \ldots, b$ and $k = 1, \ldots, n_i$.

5. For the two-factor block design with n blocks and where factor A has a levels and factor B has b levels, we choose $r = 1$, $n_1 = n$, $c = ab$ and $m_{ijk} = 1$. The index j is split into the two indices $u = 1, \ldots, a$ and $v = 1, \ldots, b$. Then $X_{uvk} \sim F_{uv}$, $k = 1, \ldots, n$.

Organization of the paper. In Section 2, asymptotic results for the general form of the nonparametric mixed model are given including a discussion on the rank transform. In Section 3, the general results are applied to the examples given above. (Note that the matched pairs design is a special case of the one-factor block design). Statistics for testing the fixed main effects and interactions in these designs are derived where also several real data sets are analysed. It is also indicated (see Section 3.4) how to derive statistics for higher-way layouts in a similar way as in the theory of linear models. Small sample approximations are discussed in Section 4 and the accuracy of the suggested approximations is analyzed by a simulation study. The proofs of the Theorems given in Section 2 are provided in the Appendix.

2 General Asymptotic Results

In this Section, the general asymptotic results for mixed models will be derived and they are applied to different designs in Section 3. Some additional notation is needed to state the asymptotic results.

Consider the general model given in (6). The vector of the distribution functions is denoted by $\mathbf{F} = (F_{11}, \ldots, F_{1c}, \ldots, F_{r1}, \ldots, F_{rc})'$ and we define $\widetilde{\mathbf{F}} = (\widetilde{F}_{11}, \ldots, \widetilde{F}_{rc})'$ where

$\widetilde{F}_{ij} = n_i^{-1} \sum_{k=1}^{n_i} \widehat{F}_{ijk}$ and $\widehat{F}_{ijk}(x) = m_{ijk}^{-1} \sum_{s=1}^{m_{ijk}} c(x - X_{ijks})$ are the empirical distribution functions within cell (i,j,k). Let $H = N^{-1} \sum_{i=1}^{r} \sum_{j=1}^{c} \sum_{k=1}^{n_i} m_{ijk} F_{ij}$ and $\widehat{H}(x) = N^{-1} \sum_{i=1}^{r} \sum_{j=1}^{c} \sum_{k=1}^{n_i} \sum_{s=1}^{m_{ijk}} c(x - X_{ijks})$ where $N = \sum_{i=1}^{r} \sum_{j=1}^{c} \sum_{k=1}^{n_i} m_{ijk}$. The vector of the generalized means $\mathbf{p} = \int H d\mathbf{F}$ is estimated by $\widetilde{\mathbf{p}} = \int \widehat{H} d\widetilde{\mathbf{F}}$. Note that the components $\widetilde{p}_{ij}$ of $\widetilde{\mathbf{p}}$ are computed from the ranks R_{ijks} of X_{ijks} among all N random variables, namely

$$\widetilde{p}_{ij} = \int \widehat{H} d\widetilde{F}_{ij} = \frac{1}{n_i} \sum_{k=1}^{n_i} \frac{1}{m_{ijk}} \sum_{s=1}^{m_{ijk}} \frac{1}{N} \left(R_{ijks} - \frac{1}{2} \right) = \frac{1}{N} \left(\widetilde{R}_{ij\cdot\cdot} - \frac{1}{2} \right) . \tag{7}$$

Let $Y_{ijks} = H(X_{ijks})$ denote the asymptotic rank transform (ART) of X_{ijks} and let $\overline{Y}_{ijk\cdot} = m_{ijk}^{-1} \sum_{s=1}^{m_{ijk}} Y_{ijks}$ denote the mean of the ART's of the replications for subject k under treatment j within group i. Let $\widetilde{Y}_{ij\cdot\cdot} = n_i^{-1} \sum_{k=1}^{n_i} \overline{Y}_{ijk\cdot}$ denote the unweighted mean of the $\overline{Y}_{ijk\cdot}$'s over the n_i subjects and let $\widetilde{\mathbf{Y}}_{i\cdot\cdot} = (\widetilde{Y}_{i1\cdot\cdot}, \ldots, \widetilde{Y}_{ic\cdot\cdot})'$ and $\widetilde{\mathbf{Y}}_{\cdot\cdot} = \left(\widetilde{\mathbf{Y}}_{1\cdot\cdot}', \ldots, \widetilde{\mathbf{Y}}_{r\cdot\cdot}' \right)'$. Finally, let $\mathbf{V}_i = Cov\left(\sqrt{N}\, \widetilde{\mathbf{Y}}_{i\cdot\cdot} \right)$ denote the covariance matrix of $\sqrt{N}\, \widetilde{\mathbf{Y}}_{i\cdot\cdot}$ and let $\rho_m(i)$ denote the smallest eigenvalue of $\mathbf{V}_i$.

The results will be derived under the following assumptions.

Assumptions.

A1. $\min n_i \to \infty$.

A2. $1 \leq m_{ijk} \leq M < \infty$, i.e. the number of replications is uniformly bounded.

A3. $0 < \lambda_0 \leq n_i/N \leq 1 - \lambda_0 < 1$.

A4. $\rho_m(i) \geq k_0 > 0$, $i = 1, \ldots, r$.

Lemma 2.1 *Let $\mathbf{X}_{ik} = (\mathbf{X}_{i1k}', \ldots, \mathbf{X}_{ick}')'$ be independent random vectors as defined in (6). Then, under assumptions A1 and A2, $\widetilde{\mathbf{p}} = \int \widehat{H} d\widetilde{\mathbf{F}}$ is consistent for $\mathbf{p} = \int H d\mathbf{F}$ in the sense that $\widetilde{\mathbf{p}} - \mathbf{p} \xrightarrow{p} \mathbf{0}$.*

Proof: see Appendix.

Theorem 2.2 *Let $\mathbf{X}_{ik}$ as in Lemma 2.1. Then, under assumptions A1 and A2,*

$$\sqrt{N} \int \widehat{H} d\left(\widetilde{\mathbf{F}} - \mathbf{F} \right) \doteq \sqrt{N} \int H d\left(\widetilde{\mathbf{F}} - \mathbf{F} \right) = \sqrt{N} \left(\widetilde{\mathbf{Y}}_{\cdot\cdot} - \mathbf{p} \right) .$$

Proof: see Appendix.

Note that the vectors $\widetilde{\mathbf{Y}}_{i\cdot\cdot}$ are independent and thus,

$$\mathbf{V} = Cov\left(\sqrt{N}\, \widetilde{\mathbf{Y}}_{\cdot\cdot} \right) = \bigoplus_{i=1}^{r} \mathbf{V}_i \tag{8}$$

which can be estimated by the ranks R_{ijks}.

Theorem 2.3 *Let* $\overline{\mathbf{R}}_{ik\cdot} = \left(\overline{R}_{i1k\cdot}, \ldots, \overline{R}_{ick\cdot}\right)'$ *where* $\overline{R}_{ijk\cdot} = m_{ijk}^{-1} \sum_{s=1}^{m_{ijk}} R_{ijks}$ *and* R_{ijks} *is the rank of* X_{ijks} *among all the* N *observations. Further let* $\widetilde{\mathbf{R}}_{i\cdot\cdot} = (\widetilde{R}_{i1\cdot\cdot}, \ldots, \widetilde{R}_{ic\cdot\cdot})'$ *where* $\widetilde{R}_{ij\cdot\cdot} = n_i^{-1} \sum_{k=1}^{n_i} \overline{R}_{ijk\cdot}$*. Let* $\mathbf{V}$ *be as given in (8) and let*

$$\widehat{\mathbf{V}} \;=\; \bigoplus_{i=1}^{r} \widehat{\mathbf{V}}_i = \bigoplus_{i=1}^{r} \frac{1}{Nn_i(n_i-1)} \sum_{k=1}^{n_i} \left(\overline{\mathbf{R}}_{ik\cdot} - \widetilde{\mathbf{R}}_{i\cdot\cdot}\right)\left(\overline{\mathbf{R}}_{ik\cdot} - \widetilde{\mathbf{R}}_{i\cdot\cdot}\right)' \;.$$

Then, under assumptions A1 - A3,

$$\left\|\widehat{\mathbf{V}} - \mathbf{V}\right\| \;\xrightarrow{p}\; 0.$$

Proof: see Appendix.

Next, the asymptotic distribution of $\sqrt{N}\mathbf{C}\widetilde{\mathbf{p}}$ under the hypothesis $H_0^F : \mathbf{CF} = \mathbf{0}$, where $\mathbf{C}$ is a contrast matrix, will be given.

Theorem 2.4 *Let* $\mathbf{X}_{ik}$ *as in Lemma 2.1 and* $\widetilde{\mathbf{Y}}_{i\cdot\cdot}$ *as in Theorem 2.2. Let* $\mathbf{C}$ *be a contrast matrix and consider the hypothesis* $H_0^F : \mathbf{CF} = \mathbf{0}$*. Then, under assumptions A1 - A4, and under* H_0^F*,*

1. *the statistic* $\sqrt{N}\mathbf{C}\widetilde{\mathbf{p}} = \sqrt{N}\mathbf{C}\int \widehat{H} d\widetilde{\mathbf{F}}$ *has asymptotically a multivariate normal distribution with mean* $\mathbf{0}$ *and covariance matrix* $\mathbf{CVC}'$*,*

2. *the quadratic form* $Q_N^*(\mathbf{C}) = N\widetilde{\mathbf{p}}'\mathbf{C}'\left[\mathbf{CVC}'\right]^{-}\mathbf{C}\widetilde{\mathbf{p}}$ *has asymptotically a central* χ_f^2*-distribution with* $f = rank(\mathbf{C})$ *where* $[\mathbf{CVC}']^{-}$ *denotes a generalized inverse of* $[\mathbf{CVC}']$*.*

3. *If* $\mathbf{C}$ *is of full row rank, then* $Q_N(\mathbf{C}) = N\widetilde{\mathbf{p}}'\mathbf{C}'\left[\mathbf{C}\widehat{\mathbf{V}}\mathbf{C}'\right]^{-1}\mathbf{C}\widetilde{\mathbf{p}}$ *has asymptotically a central* χ_f^2*-distribution with* $f = rank(\mathbf{C})$*.*

4. *Let* $\widehat{\mathbf{W}}$ *be the matrix corresponding to* $\mathbf{W}$ *given in (5) where* $\mathbf{V}$ *is replaced with* $\widehat{\mathbf{V}}$ *defined in Theorem 2.3. Then* $Q_N(\widehat{\mathbf{W}}) = N\widetilde{\mathbf{p}}'\widehat{\mathbf{W}}\widetilde{\mathbf{p}}$ *has asymptotically a central* χ_f^2*-distribution with* $f = rank(\mathbf{W})$*.*

Proof: see Appendix.

Finally, the efficacy of the test statistics will be considered. Define a sequence of alternatives

$$\mathbf{F}_N = (F_{N,11}, \ldots, F_{N,rc})' = \left(1 - \frac{1}{\sqrt{N}}\right)\mathbf{F} + \frac{1}{\sqrt{N}}\mathbf{K} \tag{9}$$

contiguous to the nonparametric null hypothesis $H_0^F : \mathbf{CF} = \mathbf{0}$ where $\mathbf{K} = (K_{11}, \ldots, K_{rc})'$ is some vector of distribution functions. Let further $\boldsymbol{\nu} = \sqrt{N}\mathbf{C}\int H d\mathbf{F}_N = \int H d(\mathbf{CK})$.

Theorem 2.5 *Let* $\mathbf{X}_{ik}$ *as in Lemma 2.1 and assume that* $X_{ijks} \sim F_{N,ij}(x) = (1 - N^{-1/2})F_{ij}(x) + N^{-1/2}K_{ij}(x)$, $i = 1, \ldots, r$, $j = 1, \ldots, c$. *Let further* $\mathbf{F} = (F_{11}, \ldots, F_{rc})'$ *in (9) and let* $\mathbf{C}$ *be some contrast matrix of full row rank such that* $\mathbf{CF} = \mathbf{0}$. *Further let* $H(x) = N^{-1}\sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{k=1}^{n_i} m_{ijk}F_{ij}(x)$ *denote the weighted average distribution function of* $F_{11}, \ldots, F_{rc}$ *and let* $H_N(x) = N^{-1}\sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{k=1}^{n_i} m_{ijk}F_{N,ij}(x)$ *denote the weighted average distribution function od* $F_{N,11}, \ldots, F_{N,rc}$. *Then under the assumptions A1 - A4 and under the sequence of alternatives* $\mathbf{F}_N$ *in (9),*

1. $\widetilde{\boldsymbol{\nu}} = \sqrt{N}\mathbf{C}\widetilde{\mathbf{p}} = \sqrt{N}\mathbf{C}\int \widehat{H}d\widetilde{\mathbf{F}}$ *has asymptotically a multivariate normal distribution with mean* $\boldsymbol{\nu}$ *and covariance matrix* $\mathbf{CVC}'$ *where* $\mathbf{V}$ *is defined in (8) and*

2. $Q_N(\mathbf{C}) = N\widetilde{\mathbf{p}}'\mathbf{C}'\left[\mathbf{CVC}'\right]^{-1}\mathbf{C}\widetilde{\mathbf{p}}$ *has asymptotically a noncentral* $\chi^2_f(\gamma)$*-distribution with* $f = rank(\mathbf{C})$ *d.f. and with noncentrality parameter* $\gamma = \boldsymbol{\nu}'\left[\mathbf{CVC}'\right]^{-1}\boldsymbol{\nu}$.

Proof: see Appendix.

It follows from this theorem that the test is consistent against alternatives of the form $\mathbf{Cp} \neq \mathbf{0}$. Note that this does not imply $\mathbf{CK} \neq \mathbf{0}$.

The 'Rank Transform' (RT) Property. The statistics given in Theorem (2.4) can formally be derived from MANOVA statistics by replacing the original observations X_{ijks} by their ranks R_{ijks}. The name 'rank transform' (RT) has been introduced for this technique in the univariate one-factor fixed model (Conover & Iman, 1976, 1981). The derivation of rank tests for more complex experimental designs by this heuristic technique may lead to incorrect procedures. First of all, the underlying testing problem has to be identified. It follows from Theorems 2.2, and 2.4 that the rank statistic $\sqrt{N}\mathbf{C}\int \widehat{H}d\widetilde{\mathbf{F}}$ and the statistic $\sqrt{N}\mathbf{C}\int Hd\widetilde{\mathbf{F}} = \sqrt{N}\mathbf{C}\widetilde{\mathbf{Y}}_{\cdot\cdot}$ are asymptotically equivalent if the hypothesis is formulated as $H_0^F : \mathbf{CF} = \mathbf{0}$, i.e. in terms of the distribution functions. A second point is that any assumed homoscedasticity of the random variables X_{ijks} is not transferred to the ART $Y_{ijks} = H(X_{ijks})$, in general. This has been pointed out by Akritas (1990). If all distribution functions are equal, i.e. $H = F_{11} = \cdots = F_{rc}$, then $\mathbf{V} = \sigma^2\mathbf{I}_r \otimes \mathbf{I}_c$ in a model with independent observations. This shows why the RT works e.g. in the one-factor fixed design under the hypothesis $H_0^F : F_1 = \cdots = F_a$.

If a statistic has the property of being a 'RT-statistic', then this is of importance for computational purposes. The parametric counterpart of a RT-statistic which may be available in a statistical software package can be applied to the ranked data. Only the quality of approximation to the asymptotic distribution or some finite approximation has to be taken care of. In any case, it is necessary to identify the statistical model of the ART under the hypothesis. The RT should not be regarded as a technique to derive statistics rather than a property of a statistic which can be useful for computational purposes.

In the next Section, the general results given here will be applied to different two- and three-way layouts.

3 Examples and Applications to Different Designs

3.1 Two-Factor Mixed Models / Cross-Classification

The nonparametric model. In a cross-classified mixed model, the random variables X_{kjs} are observed on the k-th randomly chosen subject (or block), $k = 1, \ldots, n$ which is repeatedly observed (or measured) under treatment $j = 1, \ldots, b$ and $s = 1, \ldots, m_{kj}$ repeated observations are made on the same subject k under treatment j. Thus, the general two-factor mixed model can be described by independent random vectors

$$\begin{aligned} \mathbf{X}_k &= (\mathbf{X}'_{k1}, \ldots, \mathbf{X}'_{kb})', \ k = 1, \ldots, n, \quad \text{where} \quad \mathbf{X}_{kj} = (X_{kj1}, \ldots, X_{kjm_{kj}})' \\ & \quad \text{and} \quad X_{kjs} \sim F_j(x) \ , \ k = 1, \ldots, n, \ j = 1, \ldots, b, \ s = 1, \ldots, m_{kj} \ . \end{aligned} \tag{10}$$

Treatment effects in the nonparametric model (10) will be described by the generalized means $p_j = \int H dF_j, \ j = 1, \ldots, b$ where $H = N^{-1} \sum_{j=1}^b N_j F_j$ and $N = \sum_{j=1}^b N_j = \sum_{k=1}^n \sum_{j=1}^b m_{kj}$. Let $\mathbf{F} = (F_1, \ldots, F_b)'$ then $\mathbf{p} = (p_1, \ldots, p_b)' = \int H d\mathbf{F}$.

Hypotheses. We are mainly interested in analyzing the fixed treatment effect. In the classical linear model theory it is assumed that $X_{kjs} \sim F(x - \mu_j)$ where $\mu_j = E(X_{kjs})$, $k = 1, \ldots, n, \ s = 1, \ldots, m_{kj}$. To define treatment effects and to formulate hypotheses, the expectations μ_j are considered and the hypothesis of no treatment effect is written as $H_0^\mu : \mathbf{P}_b \boldsymbol{\mu} = \mathbf{0}$ where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_b)'$ and $\mathbf{P}_b$ is given in (4). In the nonparametric model (10), the hypothesis $H_0^F : \mathbf{P}_b \mathbf{F} = \mathbf{0}$ is the same as in the one-factor fixed model and is only related to the one-dimensional marginal distributions. It is easy to see that the hypotheses H_0^F and H_0^μ are equivalent in the linear model.

Derivation of the Statistic. Nonparametric tests for this model have been considered by Thompson (1991) and Akritas & Arnold (1994). In both papers, ties were excluded and only the case of $m_{kj} = 1$ replication per cell is considered. Below, these results will be generalized to an unequal number of replications $m_{kj} \geq 1$ per treatment j and block k. Moreover, ties are allowed and a simpler form of the statistic is derived.

The notation introduced in Section 2 is adopted to the two factor mixed model. Let

$$\begin{aligned} \widetilde{\mathbf{F}} &= (\widetilde{F}_1, \ldots, \widetilde{F}_b)' \ , \quad \widetilde{F}_j(x) = \frac{1}{n} \sum_{k=1}^n \frac{1}{m_{kj}} \sum_{s=1}^{m_{kj}} c(x - X_{kjs}) \ , \\ \widehat{H}(x) &= \frac{1}{N} \sum_{k=1}^n \sum_{j=1}^b \sum_{s=1}^{m_{kj}} c(x - X_{kjs}) \ . \end{aligned}$$

The statistic will be based on the vector of unweighted means $\widetilde{\mathbf{p}} = (\widetilde{p}_1, \ldots, \widetilde{p}_b)' = \int \widehat{H} d\widetilde{\mathbf{F}}$ which is a consistent estimate of $\mathbf{p} = \int H d\mathbf{F}$. The components $\widetilde{p}_j$ are computed from the ranks R_{kjs} of X_{kjs} among all N random variables, namely

$$\widetilde{p}_j = \int \widehat{H} d\widetilde{F}_j = \frac{1}{n} \sum_{k=1}^n \frac{1}{m_{kj}} \sum_{s=1}^{m_{kj}} \frac{1}{N} \left(R_{kjs} - \frac{1}{2} \right) = \frac{1}{N} \left(\widetilde{R}_{\cdot j \cdot} - \frac{1}{2} \right) \ . \tag{11}$$

Let $\mathbf{R}_k = (\overline{R}_{k1\cdot}, \ldots, \overline{R}_{kb\cdot})'$, $k = 1, \ldots, n$ denote the vector of the rank means for subject k where $\overline{R}_{kj\cdot} = m_{kj}^{-1} \sum_{k=1}^{m_{kj}} R_{kjs}$. Let further $\widetilde{\mathbf{R}}_\cdot = n^{-1} \sum_{k=1}^{n} \mathbf{R}_k$ denote the unweighted mean of the vectors $\mathbf{R}_k$. Similarly, let $\widetilde{\mathbf{Y}}_\cdot = n^{-1} \sum_{k=1}^{n} \mathbf{Y}_k$ be the mean of $\mathbf{Y}_k = \left(\overline{Y}_{k1\cdot}, \ldots, \overline{Y}_{kb\cdot}\right)'$ where $\overline{Y}_{kj\cdot} = m_{kj}^{-1} \sum_{s=1}^{m_{kj}} Y_{kjs}$ and $Y_{kjs} = H\left(X_{kjs}\right)$ is the ART.

It follows from Theorem 2.2 that the statistics $\sqrt{n}\mathbf{C}\widetilde{\mathbf{p}} = \sqrt{n}\mathbf{C} \int \widehat{H} d\widehat{\mathbf{F}}$ and $\sqrt{n}\mathbf{C}\widetilde{\mathbf{Y}}_\cdot$ are asymptotically equivalent under $H_0^F : \mathbf{CF} = \mathbf{0}$. A consistent estimate of the covariance matrix $\mathbf{V} = Cov\left(\sqrt{n}\, \widetilde{\mathbf{Y}}_\cdot\right)$ follows directly from Theorem 2.3, namely

$$\widehat{\mathbf{V}}_n = \frac{1}{N^2(n-1)} \sum_{k=1}^{n} \left(\mathbf{R}_k - \widetilde{\mathbf{R}}_\cdot\right)\left(\mathbf{R}_k - \widetilde{\mathbf{R}}_\cdot\right)' \; . \qquad (12)$$

Note that the result of Theorem 2.2 remains true if the statistic is multiplied by $\sqrt{n}$ instead of $\sqrt{N}$.

To derive the statistic, we choose the contrast matrix $\mathbf{W} = \mathbf{V}^{-1}\left[\mathbf{I}_b - \mathbf{J}_b \mathbf{V}^{-1} / \mathbf{1}_b' \mathbf{V}^{-1} \mathbf{1}_b\right]$ and we note that $\mathbf{WF} = \mathbf{0}$ iff $\mathbf{P}_b\mathbf{F} = \mathbf{0}$ (see Lemma A.2). Moreover, it follows from Lemma A.1 that $\mathbf{WVW} = \mathbf{W}$. Under $H_0^F : \mathbf{P}_b\mathbf{F} = \mathbf{0}$, the statistics $\sqrt{n}\, \mathbf{W}\widetilde{\mathbf{p}}$ and $\sqrt{n}\, \mathbf{W}\, \widetilde{\mathbf{Y}}_\cdot$ are asymptotically equivalent and a rank statistic for testing H_0^F can be derived from the ART. Let $\widehat{\mathbf{W}} = \widehat{\mathbf{V}}_n^{-1}\left[\mathbf{I}_b - \mathbf{J}_b \widehat{\mathbf{V}}_n^{-1} / \mathbf{1}_b' \widehat{\mathbf{V}}_n^{-1} \mathbf{1}_b\right]$ where $\widehat{\mathbf{V}}_n$ is given in (12) and note that

$$\widehat{Q}(\mathbf{W}) = \sqrt{n}\left(\widehat{\mathbf{W}}\widetilde{\mathbf{p}}\right)'\left(\widehat{\mathbf{W}}\widehat{\mathbf{V}}\widehat{\mathbf{W}}\right)^{-}\left(\widehat{\mathbf{W}}\widetilde{\mathbf{p}}\right)\sqrt{n} = n\; \widetilde{\mathbf{p}}'\; \widehat{\mathbf{W}}\; \widehat{\mathbf{W}}^{-}\widehat{\mathbf{W}}\; \widetilde{\mathbf{p}} = n\; \widetilde{\mathbf{p}}'\; \widehat{\mathbf{W}}\; \widetilde{\mathbf{p}} \; .$$

Denote the (i,j)-element of $\widehat{\mathbf{V}}_n^{-1}$ by $\widehat{s}_{ij}$, $i,j = 1, \ldots, b$ and let $\widehat{s}_{\cdot j} = \sum_{i=1}^{b} \widehat{s}_{ij}$ and $\widehat{s}_{\cdot\cdot} = \sum_{j=1}^{b} \widehat{s}_{\cdot j}$. Then it follows from Theorem 2.4 that the statistic

$$\begin{aligned} Q_n^M(B) &= n\; \widetilde{\mathbf{p}}'\; \widehat{\mathbf{W}}\; \widetilde{\mathbf{p}} = \frac{n}{N^2}\left[\widetilde{\mathbf{R}}_\cdot'\; \widehat{\mathbf{V}}_n^{-1}\; \widetilde{\mathbf{R}}_\cdot - \frac{1}{\widehat{s}_{\cdot\cdot}}\left(\widetilde{\mathbf{R}}_\cdot'\; \widehat{\mathbf{V}}_n^{-1} \mathbf{1}_b\right)^2\right] \\ &= \frac{n}{N^2}\left[\sum_{i=1}^{b}\sum_{j=1}^{b} \widetilde{R}_{\cdot i \cdot} \widehat{s}_{ij} \widetilde{R}_{\cdot j \cdot} - \frac{1}{\widehat{s}_{\cdot\cdot}}\left(\sum_{j=1}^{b} \widehat{s}_{\cdot j} \widetilde{R}_{\cdot j \cdot}\right)^2\right] \end{aligned} \qquad (13)$$

has asymptotically $(n \to \infty)$ a central χ_f^2-distribution with $f = rank(\mathbf{W}) = b - 1$ d.f. under $H_0^F : \mathbf{P}_b\mathbf{F} = \mathbf{0}$. Note that $Q_n^M(B)$ has the RT-property with respect to a parametric statistic for a repeated measurements model with an unspecified covariance structure and it can be computed from the ranks by any appropriate statistical software package. For small samples, the statistic $(n - b + 1)Q_n^M(B)/[(b-1)(n-1)]$ may be approximated by a central F-distribution with $f_1 = b - 1$ and $f_2 = n - b + 1$ d.f. (see Section 4).

Example 1. The test derived in this Section, will be applied to the probe word data given by Timm (1980) where each of 11 subjects is given five probe words and the reaction time is measured. The ranks of the original data and the rank means $\overline{R}_{\cdot j}$ for the $j = 1, \ldots, 5$ probe words are displayed in Table 1. The result is $Q_n^M(B) = 24.32$ and since the sample size is small, $7Q_n^M(B)/40 = 4.256$ is compared with the F-distribution with $f_1 = 4$ and $f_2 = 7$ d.f. resulting a p-value of $p = 0.046$.

Here insert Table 1.

3.2 Two-Factor Mixed Models / Nested Designs

The nonparametric model. In the mixed model where the random factor B is nested under the fixed factor A, the observations X_{iks}, $i = 1, \ldots, a$, $k = 1, \ldots, n_i$, $s = 1, \ldots, m_{ik}$ are made on $N = \sum_{i=1}^{a} n_i$ randomly chosen subjects which are repeatedly ($s = 1, \ldots, m_{ik}$) measured under the same treatment i. The random variables X_{iks} and $X_{i'k's'}$ are independent if $i \neq i'$ or if $k \neq k'$ where X_{iks} and $X_{ik's'}$ are identically distributed according to $F_i(x)$. Note that the random variables X_{iks} and $X_{iks'}$ may be dependent. Thus, the nonparametric model may be written by independent random vectors

$$\mathbf{X}_{ik} = (X_{ik1}, \ldots, X_{ikm_{ik}})', \; i = 1, \ldots, a, \; k = 1, \ldots, n_i \qquad (14)$$

where $X_{iks} \sim F_i(x)$, $k = 1, \ldots, n_i$, $s = 1, \ldots, m_{ik}$.

Hypotheses. In the linear model, the expectations $\mu_i = E(X_{iks})$, $k = 1, \ldots, n_i$, $s = 1, \ldots, m_{ik}$ are considered and the hypothesis of no treatment effect is formulated as $H_0^\mu : \mathbf{P}_a\boldsymbol{\mu} = \mathbf{0}$ where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_a)'$. In the nonparametric model, the hypothesis is analogously formulated as $H_0^F : \mathbf{P}_a\mathbf{F} = \mathbf{0}$ where $\mathbf{F} = (F_1, \ldots, F_a)'$. It is obvious that the hypotheses H_0^F and H_0^μ are equivalent in the linear model.

Derivation of the Statistic. Here, like in the case of the cross-classification, unequal numbers m_{ik} of replications for subject k under treatment i are admitted. The statistic will be based on the generalized mean vector $\mathbf{p} = (p_1, \ldots, p_a)' = \int H d\mathbf{F}$ where $H = N^{-1}\sum_{i=1}^{a}\sum_{k=1}^{n_i} m_{ik}F_i$ and $N = \sum_{i=1}^{a}\sum_{k=1}^{n_i} m_{ik}$. Let $\widetilde{\mathbf{F}} = (\widetilde{F}_1, \ldots, \widetilde{F}_a)'$ where $\widetilde{F}_i = n_i^{-1}\sum_{k=1}^{n_i} m_{ik}^{-1}\sum_{s=1}^{m_{ik}} c(x - X_{iks})$. Then the components $p_i = \int H dF_i$ of $\mathbf{p}$ are estimated by an unweighted sum of cell means $\widetilde{p}_i = \int \widehat{H} d\widetilde{F}_i = N^{-1}(\widetilde{R}_{i\cdot\cdot} - 1/2)$ where $\widetilde{R}_{i\cdot\cdot} = n_i^{-1}\sum_{k=1}^{n_i}\overline{R}_{ik\cdot}$ and $\overline{R}_{ik\cdot} = m_{ik}^{-1}\sum_{s=1}^{m_{ik}} R_{iks}$. Here, R_{iks} is the rank of X_{iks} among all $N = \sum_{i=1}^{a}\sum_{k=1}^{n_i} m_{ik}$ observations.

Let $\widetilde{\mathbf{Y}}_{\cdot} = (\widetilde{Y}_{1\cdot\cdot}, \ldots, \widetilde{Y}_{a\cdot\cdot})' = \int H d\widetilde{\mathbf{F}}$ where $\widetilde{Y}_{i\cdot\cdot} = n_i^{-1}\sum_{k=1}^{n_i}\overline{Y}_{ik\cdot} = n_i^{-1}\sum_{k=1}^{n_i} m_{ik}^{-1}\sum_{s=1}^{m_{ik}} Y_{iks}$ and $Y_{iks} = H(X_{iks})$ is the ART. It follows from Theorem 2.2 that the statistics $\sqrt{N}\mathbf{C}\int \widehat{H} d\widetilde{\mathbf{F}}$ and $\sqrt{N}\mathbf{C}\int H d\widetilde{\mathbf{F}} = \sqrt{N}\mathbf{C}\widetilde{\mathbf{Y}}_{\cdot}$ are asymptotically equivalent under $H_0^F : \mathbf{CF} = \mathbf{0}$.

Note that the random variables $\overline{Y}_{ik\cdot}$ are independent and thus, $\mathbf{V}_a = Cov(\sqrt{N}\,\widetilde{\mathbf{Y}}_{\cdot}) = \bigoplus_{i=1}^{a}\frac{N}{n_i}\sigma_i^2$ where $\sigma_i^2 = n_i^{-1}\sum_{k=1}^{n_i}\sigma_{ik}^2$ and $\sigma_{ik}^2 = Var(\overline{Y}_{ik\cdot})$. To estimate the variances σ_i^2 consistently, let $S_i^2 = \sum_{k=1}^{n_i}(\overline{R}_{ik\cdot} - \widetilde{R}_{i\cdot\cdot})^2$ and $\widehat{\sigma}_i^2 = [N^2(n_i - 1)]^{-1}S_i^2$.

To derive a statistic for testing $H_0^F : \mathbf{CF} = \mathbf{0}$, let $\mathbf{W} = \mathbf{V}_a^{-1}(\mathbf{I}_a - \mathbf{J}_a\mathbf{V}_a^{-1}/trace(\mathbf{V}_a^{-1}))$, $\widehat{\mathbf{V}}_a = \bigoplus_{i=1}^{a}\frac{N}{n_i}\widehat{\sigma}_i^2$ and $\widehat{\mathbf{W}} = \widehat{\mathbf{V}}_a^{-1}(\mathbf{I}_a - \mathbf{J}_a\widehat{\mathbf{V}}_a^{-1}/trace(\widehat{\mathbf{V}}_a^{-1}))$. Then $\widehat{\mathbf{W}}\,\widehat{\mathbf{V}}_a\widehat{\mathbf{W}} = \widehat{\mathbf{W}}$ and $\mathbf{WF} = \mathbf{0}$ iff $\mathbf{P}_a\mathbf{F} = \mathbf{0}$. It follows from Theorem 2.4 that the quadratic form

$$Q_N^H = N\widetilde{\mathbf{p}}'\,\widehat{\mathbf{W}}\,\widetilde{\mathbf{p}} = \sum_{i=1}^{a}\frac{n_i(n_i-1)}{S_i^2}\left(\widetilde{R}_{i\cdot\cdot} - \frac{1}{\sum_{r=1}^{a}(n_r/\widehat{\sigma}_r^2)}\sum_{r=1}^{a}\frac{n_r\widetilde{R}_{r\cdot\cdot}}{\widehat{\sigma}_r^2}\right)^2 \qquad (15)$$

has asymptotically ($n_i \to \infty$) a χ^2_f-distribution with $f = a-1$ d.f. under $H_0^F : \mathbf{P}_a\mathbf{F} = \mathbf{0}$. For small samples, the null distribution of the statistic $Q_N^H/(a-1)$ may be approximated by the F-distribution with $f_1 = a-1$ and $f_2 = n. - a$ d.f. where $n. = \sum_{i=1}^a n_i$. Note that Q_N^H has the RT-property with respect to a parametric statistic for the hierarchical mixed model and it can be computed from the ranks by any appropriate statistical software package.

In case of an equal number of replications $m_{ik} \equiv m$, $\overline{R}_{ik\cdot} = m^{-1}\sum_{s=1}^m R_{iks}$, $\widetilde{R}_{i\cdot\cdot} = (mn_i)^{-1}R_{i\cdot\cdot} = \overline{R}_{i\cdot\cdot}$, $\widetilde{R} = \overline{R}_{i\cdot\cdot} = (N+1)/2$ and $\widehat{\sigma}^2 = (n.-a)^{-1}\sum_{i=1}^a\sum_{k=1}^{n_i}(\overline{R}_{ik\cdot} - \overline{R}_{i\cdot\cdot})^2$. In this case, the quadratic form Q_N^H given in (15) reduces to

$$Q_N^H = (n.-a)\cdot\frac{\sum_{i=1}^a n_i\left(\overline{R}_{i\cdot\cdot} - (N+1)/2\right)^2}{\sum_{i=1}^a\sum_{k=1}^{n_i}\left(\overline{R}_{ik\cdot} - \overline{R}_{i\cdot\cdot}\right)^2}$$

which has been given by Brunner and Neumann (1982) for the case of no ties.

Example 2.

We analyze the data set given by Brunner (1991) where the surface-to-volume ratio of the mitochondria in the AV-nodes in the hearts of dogs under two treatments is observed three times for each dog and for 5 dogs per treatment. For the data set and the description of the trial, we refer to Brunner (1991). Note that in this example, $m_{ik} \equiv 2$ and thus, $\sigma_i^2 = \sigma^2$, $i = 1,2$ under H_0^F. The result is $Q_N^H = 47.65$ and since the sample size is small, Q_N^H is compared with an F-distribution with $f_1 = a-1 = 1$ and $f_2 = n. - a = 8$ d.f. which yields $p = 0.00012$.

3.3 Three-factor Mixed Models with Two Factors Fixed

3.3.1 Partially Nested Designs

The nonparametric model. In this section, we consider two-factor repeated measurements designs where the repeated measurements are only taken on one factor, factor B with $j = 1,\ldots,b$ levels say, and the subjects are nested within the $i = 1,\ldots,a$ levels of factor A. Therefore, this design is called 'partially nested' design. In medical and psychological studies it appears either when different groups of subjects are observed under the same treatments for each subject or when subjects are divided randomly into several treatment groups and the outcomes are observed sequentially at several time points.

Nonparametric procedures for this design have been considered by Brunner & Neumann (1984, 1986a) for the 2×2 design with heteroscedastic distributions. Designs with $a, b \geq 2$ levels have been considered by Thompson & Ammann (1990), Thompson (1991) and Akritas (1993). Rank tests for some joint hypotheses are derived in these papers which will be explained below. First, the model shall be stated.

For the partially nested design, the nonparametric model is derived from the general mixed model (6) by letting $r = a$ and $c = b$ and is formulated by independent vectors

$$\mathbf{X}_{ik} = (\mathbf{X}'_{i1k}, \ldots, \mathbf{X}'_{ibk})', \quad i = 1, \ldots, a, \; k = 1, \ldots, n_i \tag{16}$$

where $\mathbf{X}'_{ijk} = (X_{ijk1}, \ldots, X_{ijkm_{ijk}})'$ and $X_{ijks} \sim F_{ij}(x)$, $k = 1, \ldots, n_i$, $s = 1, \ldots, m_{ijk}$.

Hypotheses. The hypotheses are formulated in terms of the marginal distribution functions F_{ij}. Let $\mathbf{F} = (F_{11}, \ldots, F_{ab})'$, then the hypotheses are:

$$\begin{array}{lll}
H_0^F(A): & (\mathbf{P}_a \otimes \frac{1}{b}\mathbf{1}'_b)\mathbf{F} = & \mathbf{0}, \text{ (no main group effect)}, \\
H_0^F(B): & (\frac{1}{a}\mathbf{1}'_a \otimes \mathbf{P}_b)\mathbf{F} = & \mathbf{0}, \text{ (no main treatment effect)}, \\
H_0^F(AB): & (\mathbf{P}_a \otimes \mathbf{P}_b)\mathbf{F} \; = & \mathbf{0}, \text{ (no interaction)}, \\
H_0^F(A|B): & (\mathbf{P}_a \otimes \mathbf{I}_b)\mathbf{F} \; = & \mathbf{0}, \text{ (no simple group effect within the treatments)}, \\
H_0^F(B|A): & (\mathbf{I}_a \otimes \mathbf{P}_b)\mathbf{F} \; = & \mathbf{0}, \text{ (no simple treatment effect within the groups)} .
\end{array}$$

The common hypotheses for the linear model are formulated in the same way by replacing the vector $\mathbf{F}$ of the distribution functions by the vector $\boldsymbol{\mu} = \int x d\mathbf{F}$ of the means. Some relations between the hypotheses stated above are given in the next Lemma.

Lemma 3.1

1. *In the nonparametric model (16),*

$$H_0^F(A|B) \Rightarrow H_0^F(A), \; H_0^F(B|A) \Rightarrow H_0^F(B).$$

2. *In the linear model,*

$$\begin{array}{l}
H_0^F(A) \Rightarrow H_0^\mu(A), \; H_0^F(B) \Rightarrow H_0^\mu(B), \; H_0^F(AB) \Rightarrow H_0^\mu(AB), \\
H_0^F(A|B) \Longleftrightarrow H_0^\mu(A|B), \; H_0^F(B|A) \Longleftrightarrow H_0^\mu(B|A). \\
\textit{If} \quad (\mathbf{P}_a \otimes \mathbf{P}_b)\boldsymbol{\mu} = \mathbf{0}, \quad \textit{then} \quad H_0^F(A) \Longleftrightarrow H_0^\mu(A) \quad \textit{and} \quad H_0^F(B) \Longleftrightarrow H_0^\mu(B).
\end{array}$$

The proofs of these statements are obvious and therefore omitted. For details, we refer to Brunner & Puri (1996). For the interpretation of the nonparametric hypotheses it may be noted that for the main effects the linear hypotheses and the nonparametric hypotheses are equivalent if there are no linear interactions, i.e. if the linear main effects are well defined.

Note that the model is not symmetric in the factors A and B since the subjects are nested under the groups. Therefore, two hypotheses for the simple factor effects are stated. We like to point out that in the general model no special pattern of the covariances between the observations X_{ijks} and $X_{ij'ks'}$ within one subject k is assumed.

For the general (multivariate) model, Thompson (1991) derived a rank test for the simple treatment effect, i.e. a test for $H_0^F(B|A)$ and Akritas (1993) derived rank tests for both simple factor effects, i.e. for $H_0^F(A|B)$ and for $H_0^F(B|A)$. In both papers, ties were excluded. In what follows, we will derive rank tests for the nonparametric hypotheses $H_0^F(A)$, $H_0^F(B)$, $H_0^F(A|B)$ and $H_0^F(B|A)$ from the unified approach in Section 2 and we do not assume that the distribution functions are continuous.

Estimators and notation. The statistics will be based on the vector of the generalized means $\mathbf{p} = (p_{11}, \ldots, p_{ab})' = \int H d\mathbf{F}$ where $\mathbf{F} = (F_{11}, \ldots, F_{ab})'$, $i = 1, \ldots, a$ and $j = 1, \ldots, b$. We will use the notation introduced in Section 2 putting $r = a$, $c = b$ and define $\widetilde{\mathbf{R}}_{\cdot\cdot} = (\widetilde{\mathbf{R}}'_{1\cdot\cdot}, \ldots, \widetilde{\mathbf{R}}'_{a\cdot\cdot})' = (\widetilde{R}_{11\cdot\cdot}, \ldots, \widetilde{R}_{ab\cdot\cdot})'$ and $\widetilde{\mathbf{R}}_{\cdot\cdot\cdot} = (\widetilde{R}_{\cdot 1\cdot\cdot}, \ldots, \widetilde{R}_{\cdot b\cdot\cdot})'$ where $\widetilde{R}_{\cdot j\cdot\cdot} = n_{\cdot}^{-1} \sum_{i=1}^{a} \sum_{k=1}^{n_i} \overline{R}_{ijk\cdot}$ and $n_{\cdot} = \sum_{i=1}^{a} n_i$.

Let $\mathbf{C}$ be any suitable contrast matrix, then under $H_0^F : \mathbf{CF} = \mathbf{0}$, the statistics $\sqrt{N}\mathbf{C} \int \widehat{H} d\widetilde{\mathbf{F}} = N^{-1/2}\mathbf{C}\widetilde{\mathbf{R}}_{\cdot\cdot}$ and $\sqrt{N}\mathbf{C} \int H d\widetilde{\mathbf{F}} = \sqrt{N}\mathbf{C}\widetilde{\mathbf{Y}}_{\cdot\cdot}$ are asymptotically equivalent which follows from Theorem 2.2. A consistent estimate of the covariance matrix $\mathbf{V}_i = Cov(\sqrt{N}\, \widetilde{\mathbf{Y}}_{i\cdot\cdot})$ is given in Theorem 2.3. We will use the contrast matrix

$$\mathbf{W}_i \;=\; \mathbf{V}_i^{-1} \left(\mathbf{I}_b - \mathbf{J}_b \mathbf{V}_i^{-1} / \mathbf{1}_b' \mathbf{V}_i^{-1} \mathbf{1}_b'\right) \tag{17}$$

and the corresponding matrix to $\mathbf{W}_i$ where $\mathbf{V}_i^{-1}$ is replaced by $\widehat{\mathbf{V}}_i^{-1}$ will be denoted by

$$\widehat{\mathbf{W}}_i \;=\; \widehat{\mathbf{V}}_i^{-1} \left(\mathbf{I}_b - \mathbf{J}_b \widehat{\mathbf{V}}_i^{-1} / \widehat{s}_{\cdot\cdot}^{(i)}\right) \tag{18}$$

where $\widehat{s}_{\cdot\cdot}^{(i)} = \sum_{j'=1}^{b} \widehat{s}_{\cdot j'}^{(i)} = \sum_{j=1}^{b} \sum_{j'=1}^{b} \widehat{s}_{jj'}^{(i)}$ and $\widehat{s}_{jj'}^{(i)}$ is the (j, j')-element of $\widehat{\mathbf{V}}_i^{-1}$. Note that $\mathbf{W}_i \mathbf{V}_i \mathbf{W}_i = \mathbf{W}_i$ and $\mathbf{W}_i \mathbf{F}_i = \mathbf{0}$ iff $\mathbf{P}_b \mathbf{F}_i = \mathbf{0}$, $i = 1, \ldots, a$ where $\mathbf{F}_i = (F_{i1}, \ldots, F_{ib})'$. Further notation will be explained along with the different test statistics which are given below. The derivation of theses statistics follows from the results given in section 2.

Statistics

Test for the Treatment Effect B

Hypothesis: $H_0^F(B) : \left(\frac{1}{a}\mathbf{1}_a' \otimes \mathbf{P}_b\right) \mathbf{F} = \mathbf{0}$.
Notation:
$\mathbf{V} = Cov(\sqrt{N} \left(\frac{1}{a}\mathbf{1}_a' \otimes \mathbf{I}_b\right) \widetilde{\mathbf{Y}}_{\cdot\cdot}) = a^{-2} \sum_{i=1}^{a} \mathbf{V}_i$ and $\widehat{\mathbf{V}} = a^{-2} \sum_{i=1}^{a} \widehat{\mathbf{V}}_i$ where $\widehat{\mathbf{V}}_i$ is given in Theorem 2.3. Let $\mathbf{W} = \mathbf{V}^{-1}(\mathbf{I}_b - \mathbf{J}_b \mathbf{V}^{-1} / \mathbf{1}_b' \mathbf{V}^{-1} \mathbf{1}_b)$ and $\widehat{\mathbf{W}} = \widehat{\mathbf{V}}^{-1}(\mathbf{I}_b - \mathbf{J}_b \widehat{\mathbf{V}}^{-1} / \widehat{s}_{\cdot\cdot})$ where $\widehat{s}_{\cdot\cdot} = \sum_{j=1}^{b} \sum_{j'=1}^{b} \widehat{s}_{jj'}$ and $\widehat{s}_{jj'}$ is the (j, j')-element of $\widehat{\mathbf{V}}^{-1}$.
Statistic:

$$\begin{aligned} Q_N(B) &= \frac{1}{N} \widetilde{\mathbf{R}}_{\cdot\cdot\cdot}' \; \widehat{\mathbf{W}} \; \widetilde{\mathbf{R}}_{\cdot\cdot\cdot} \\ &= \frac{1}{N} \left[\sum_{j=1}^{b} \sum_{j'=1}^{b} \widetilde{R}_{\cdot j\cdot\cdot} \; \widehat{s}_{jj'} \; \widetilde{R}_{\cdot j'\cdot\cdot} - \frac{1}{\widehat{s}_{\cdot\cdot}} \left(\sum_{j=1}^{b} \widehat{s}_{j\cdot} \widetilde{R}_{\cdot j\cdot\cdot} \right)^2 \right] \end{aligned} \tag{19}$$

has asymptotically a χ_f^2-distribution with $f = b - 1$ d.f. and $Q_N(B)$ has the RT-property with respect to a parametric statistic in a repeated measurements model with an unspecified covariance structure.

Test for the Group Effect A

Hypothesis: $H_0^F(A) : (\mathbf{P}_a \otimes \frac{1}{b}\mathbf{1}_b')\mathbf{F} = \mathbf{0}$.
Notation:

$$\begin{aligned} \tau_i^2 &= \frac{N}{n_i}\sigma_i^2 \ , \quad \sigma_i^2 = Var(\widetilde{Y}_{i\cdot k\cdot}) \ , \quad \widetilde{Y}_{i\cdot k\cdot} = b^{-1}\sum_{j=1}^{b}\overline{Y}_{ijk\cdot}, \\ \widehat{\tau}_i^2 &= \frac{1}{Nn_i(n_i-1)}S_i^2 \ , \quad S_i^2 = \sum_{k=1}^{n_i}(\widetilde{R}_{i\cdot k\cdot} - \widetilde{R}_{i\cdots})^2 \ , \quad \widetilde{R}_{i\cdots} = \frac{1}{bn_i}\sum_{k=1}^{n_i}\sum_{j=1}^{b}\overline{R}_{ijk\cdot} \ . \end{aligned}$$

Let $\mathbf{V} = Cov\left(\sqrt{N}\left(\mathbf{I}_a \otimes \frac{1}{b}\mathbf{1}_b'\right)\widetilde{\mathbf{Y}}_{\cdot\cdot}\right) = diag\{\tau_1^2, \ldots, \tau_a^2\}$ and $\mathbf{W} = \mathbf{V}^{-1}(\mathbf{I}_a - \mathbf{J}_a\mathbf{V}^{-1}/\sum_{j=1}^{a}[1/\tau_j^2])$ and let $\widehat{\mathbf{W}}$ be the matrix corresponding to $\mathbf{W}$ where τ_i^2 is replaced by $\widehat{\tau}_i^2$.
Statistic:

$$\begin{aligned} Q_N(A) &= \frac{1}{N}(\widetilde{R}_{1\cdots}, \ldots, \widetilde{R}_{a\cdots})\ \widehat{\mathbf{W}}\ (\widetilde{R}_{1\cdots}, \ldots, \widetilde{R}_{a\cdots})' \\ &= \sum_{i=1}^{a}\frac{n_i(n_i-1)}{S_i^2}\left(\widetilde{R}_{i\cdots} - \frac{1}{\sum_{r=1}^{a}[n_r(n_r-1)/S_r^2]}\sum_{r=1}^{a}\frac{n_r(n_r-1)\widetilde{R}_{r\cdots}}{S_r^2}\right)^2 \qquad (20) \end{aligned}$$

has asymptotically a χ_f^2-distribution with $f = a-1$ d.f. and $Q_N(A)$ has the RT-property with respect to a parametric statistic in a repeated measurements model where $\mathbf{V}$ is a diagonal matrix.

Test for the Interaction AB

Hypothesis: $H_0^F(AB) : F_{ij} = \overline{F}_{i\cdot} + \overline{F}_{\cdot j} - \overline{F}_{\cdot\cdot}$ or equivalently $H_0^F(AB) : (\mathbf{C}_A \otimes \mathbf{C}_B)\mathbf{F} = \mathbf{0}$ where $\mathbf{C}_A = (\mathbf{1}_{a-1} \vdots -\mathbf{I}_{a-1})$ and $\mathbf{C}_B = (\mathbf{1}_{b-1} \vdots -\mathbf{I}_{b-1})$ which are of full row rank.
Notation:
Let $\mathbf{V} = Cov\left(\sqrt{N}\ \widetilde{\mathbf{Y}}_{\cdot\cdot}\right) = \bigoplus_{i=1}^{a}\mathbf{V}_i$ and $\widehat{\mathbf{V}} = \bigoplus_{i=1}^{a}\widehat{\mathbf{V}}_i$ where $\widehat{\mathbf{V}}_i$ is given in Theorem 2.3.
Statistic:

$$Q_N(AB) = \frac{1}{N}\widetilde{\mathbf{R}}_{\cdot\cdot}'(\mathbf{C}_A' \otimes \mathbf{C}_B')\left[(\mathbf{C}_A \otimes \mathbf{C}_B)\bigoplus_{i=1}^{a}\widehat{\mathbf{V}}_i(\mathbf{C}_A' \otimes \mathbf{C}_B')\right]^{-1}(\mathbf{C}_A \otimes \mathbf{C}_B)\widetilde{\mathbf{R}}_{\cdot\cdot} \qquad (21)$$

has a central χ_f^2-distribution with $f = (a-1)(b-1)$ d.f. under $H_0^F(AB)$ and $Q_N(AB)$ has the RT-property with respect to a parametric statistic in a repeated measurements model with an unspecified covariance structure.

Test for $H_0^F(B|A)$

Hypothesis: $H_0^F(B|A) : F_{i1} = \cdots = F_{ib} = \overline{F}_{i\cdot}$ or equivalently $H_0^F(B|A) : (\mathbf{I}_a \otimes \mathbf{P}_b)\mathbf{F} = \mathbf{0}$.
Notation:

Let $\mathbf{V} = \bigoplus_{i=1}^{a} \mathbf{V}_i$ and $\mathbf{W} = \bigoplus_{i=1}^{a} \mathbf{W}_i$ where $\mathbf{W}_i$ is given in (17). $\mathbf{V}_i$ is estimated by $\widehat{\mathbf{V}}_i$ given in Theorem 2.3 and $\mathbf{W}_i$ is estimated by $\widehat{\mathbf{W}}_i$ given in (18).
Statistic:

$$Q_N(B|A) \;=\; \frac{1}{N}\widetilde{\mathbf{R}}'_{\cdot\cdot}\;\widehat{\mathbf{W}}\;\widetilde{\mathbf{R}}_{\cdot\cdot} = \frac{1}{N}\sum_{i=1}^{a}\left[\sum_{j=1}^{b}\sum_{j'=1}^{b}\widetilde{R}_{ij\cdot\cdot}\widehat{s}_{jj'}^{(i)}\widetilde{R}_{ij'\cdot\cdot} - \frac{1}{\widehat{s}_{\cdot\cdot}^{(i)}}\left(\sum_{j=1}^{b}\widetilde{R}_{ij\cdot\cdot}\widehat{s}_{j\cdot}^{(i)}\right)^2\right] \qquad (22)$$

has asymptotically a χ^2_f-distribution with $f = a(b-1)$ d.f. under $H_0^F(B|A)$. The quantities $\widehat{s}_{jj'}^{(i)}$ are given along with (18). Note that $Q_N(B|A)$ has the RT-property with respect to a parametric statistic in a repeated measurements model with an unspecified covariance structure.

Test for $H_0^F(A|B)$

Hypothesis: $H_0^F(A|B) : F_{1j} = \cdots = F_{aj} = \overline{F}_{\cdot j}$ or equivalently $H_0^F(A|B) : (\mathbf{P}_a \otimes \mathbf{I}_b)\mathbf{F} = \mathbf{0}$.
Notation:
Let $\mathbf{V} = \mathbf{I}_a \otimes \mathbf{V}_0$ where $\mathbf{V}_0 = \mathbf{V}_i$, $i = 1, \ldots, a$ under H_0^F. Let $\widehat{\mathbf{V}}_0 = a^{-1}\sum_{i=1}^{a}\widehat{\mathbf{V}}_i$ where $\widehat{\mathbf{V}}_i$ is given in Theorem 2.3. Let $\mathbf{W} = \mathbf{P}_a \otimes \mathbf{V}_0^{-1}$ and let $\widehat{s}_{jj'}$ be the (j, j')-element of $\widehat{\mathbf{V}}_0^{-1}$.

Statistic:

$$\begin{aligned} Q_N(A|B) \;&=\; \frac{1}{N}\widetilde{\mathbf{R}}'_{\cdot\cdot}\;\left[\mathbf{P}_a \otimes \widehat{\mathbf{V}}_0^{-1}\right]\widetilde{\mathbf{R}}_{\cdot\cdot} = \frac{1}{N}\sum_{i=1}^{a}\left(\widetilde{\mathbf{R}}'_{i\cdot\cdot} - \widetilde{\mathbf{R}}_{\cdot\cdot\cdot}\right)'\;\widehat{\mathbf{V}}_0^{-1}\;\left(\widetilde{\mathbf{R}}_{i\cdot\cdot} - \widetilde{\mathbf{R}}_{\cdot\cdot\cdot}\right) \\ &=\; \frac{1}{N}\sum_{i=1}^{a}\left[\sum_{j=1}^{b}\sum_{j'=1}^{b}(\widetilde{R}_{ij\cdot\cdot} - \widetilde{R}_{\cdot j\cdot\cdot})\;\widehat{s}_{jj'}\;(\widetilde{R}_{ij'\cdot\cdot} - \widetilde{R}_{\cdot j'\cdot\cdot})\right] \end{aligned} \qquad (23)$$

has asymptotically a χ^2_f-distribution with $f = b(a-1)$ d.f. under $H_0^F(A|B)$ and $Q_N(A|B)$ has the RT-property with respect to a parametric statistic in a repeated measurements model with an unspecified structure of the covariance matrix $\mathbf{V}_0$.

Example 3.

We apply the procedures derived in this Section to the data given by Zerbe (1979) where for 13 control and 20 obese patients plasma inorganic phosphate (PIP) was measured $0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, 3, 4$ and 5 hours after a standard dose oral glucose challenge. For the description of the trial and the data set, we refer to Zerbe (1979). An adequate model for this trial is the 3-factor mixed model with two crossed fixed factors, namely the factor 'group' (control, obese) and the factor 'time' (8 fixed time points). The 33 subjects are the levels of the random factor and are nested under the factor 'group'. Since PIP is 8 times repeatedly measured during 5 hours, a multivariate model is appropriate. Two standard questions are commonly to be answered for such a trail: (1) Do the two time curves have the same shape? - (2) Is there any influence of the factor time?

The first question can be answered by testing the interaction (21) between the two fixed factors. The time effect is analyzed by testing the treatment effect B (19). Since both quadratic forms have the RT-property with respect to a parametric repeated measurements model with an unspecified covariance structure, they can be easily computed from the overall ranks by any suitable statistical software package. The rank means for the PIP data are given in Table 2.

Here insert Table 2.

The results are $Q_N(AB) = 60.29$ and $Q_N(B) = 329.44$. For small sample sizes, the null distribution of $(n_1 + n_2 - b)Q_N(AB)/[(b-1)(n_1 + n_2 - 2)] = 6.946$ is approximated by an F-distribution with $f_1 = b - 1 = 7$ and $f_2 = n_1 + n_2 - b = 25$ d.f. (see Section 4) which yields $p = 0.00012$ and a highly significant different pattern for the two time curves is detected. Similarly, $25Q_N(B)/217 = 37.95$ is compared with the same F-distribution which yields $p < 10^{-5}$ and a highly significant influence of the time on PIP is detected.

3.3.2 Cross-Classified Designs

The nonparametric model. Here we consider the case where each randomly chosen subject k receives all level combinations (i, j), $i = 1, \ldots, a$, $j = 1, \ldots, b$ of two treatments A and B. The vector of all ab factor level combinations for subject k is denoted by

$$\begin{aligned} \mathbf{X}_k &= (\mathbf{X}'_{11k}, \ldots, \mathbf{X}'_{abk})', \ k = 1, \ldots, n \quad \text{where} \quad \mathbf{X}_{ijk} = (X_{ijk1}, \ldots, X_{ijkm_{ijk}})' , \\ & \quad X_{ijks} \sim F_{ij}(x), \ i = 1, \ldots, a, \ j = 1, \ldots, b, \ k = 1, \ldots, n, \ s = 1, \ldots, m_{ijk} \end{aligned} \tag{24}$$

and the vectors $\mathbf{X}_k$ are assumed to be independent. The assumption that the covariance matrix of $\mathbf{X}_k$ in the nonparametric model (24) is not changed under the treatment seems to be unrealistic. Therefore, the multivariate model with an arbitrary dependence structure of the observations within one subject seems to be the only reasonable assumption for the covariance matrix in the nonparametric model (24).

Rank tests for the two-way multivariate model have been considered by Thompson (1991) and Akritas & Arnold (1994) for the case of $m_{ijk} \equiv 1$ replication per subject and treatment combination (i, j) and for continuous distribution functions. The first named author considered a rank test for the hypothesis $H_0 : F_{ij} = F_i$, $j = 1, \ldots, b$, $i = 1, \ldots, a$ which is in fact the hypothesis $H_0(B|A) : (\mathbf{I}_a \otimes \mathbf{P}_b)\mathbf{F} = \mathbf{0}$ where $\mathbf{F} = (F_{11}, \ldots, F_{ab})'$. Note that this hypothesis is a 'joint hypothesis', i.e. main and interaction effects are tested together. Such joint hypotheses are useful in practice and have been considered already by Koch (1970) in the context of a complex split-plot design. Akritas & Arnold (1994) stated the hypotheses of the linear two-way layout model in terms of distribution functions $H_0^F(A) : (\mathbf{P}_a \otimes \frac{1}{b}\mathbf{1}'_b)\mathbf{F} = \mathbf{0}$, $H_0^F(AB) : (\mathbf{P}_a \otimes \mathbf{P}_b)\mathbf{F} = \mathbf{0}$ and $H_0^F(A|B) : (\mathbf{P}_a \otimes \mathbf{I}_b)\mathbf{F} = \mathbf{0}$, i.e. they replaced simply the mean vector $\boldsymbol{\mu} = (\mu_{11}, \ldots, \mu_{ab})'$ in the linear model by the vector of distribution functions $\mathbf{F} = (F_{11}, \ldots, F_{ab})'$ and based the statistics on a consistent estimate of $\mathbf{p} = \int H d\mathbf{F}$ where $H = N^{-1} \sum_{i=1}^a \sum_{j=1}^b M_{ij} F_{ij}$, $M_{ij} = \sum_{k=1}^n m_{ijk}$

and $N = \sum_{i=1}^a \sum_{j=1}^b M_{ij}$. In this setup, it is possible to test nonparametric main effects, interactions and simple factor effects separately in the general model (24).

Below, the results of Thompson (1991) and Akritas & Arnold (1994) will be generalized to unbalanced designs ($m_{ijk} \geq 1$). The statistics follow from the general approach derived in Section 2 and are also valid in the case of ties.

Hypotheses. The hypotheses for the three-way mixed model with two crossed fixed factors as well as the relations between the hypotheses in the different models are the same as for the partially nested design discussed in subsection 3.3.1 since the formulation of the hypotheses does not imply the structure of the covariance matrix of $\mathbf{X}_k$.

Notation. As in the previous section, we will base the statistics on a consistent estimate of the vector of the generalized means $\mathbf{p} = \int H d\mathbf{F}$ which is estimated consistently by $\tilde{\mathbf{p}} = \int \widehat{H} d\tilde{\mathbf{F}}$ where $\tilde{\mathbf{F}} = (\widetilde{F}_{11}, \ldots, \widetilde{F}_{ab})'$ and $\widetilde{F}_{ij}(x) = n^{-1} \sum_{k=1}^n \widehat{F}_{ijk}(x) = n^{-1} \sum_{k=1}^n \sum_{s=1}^{m_{ijk}} c(x - X_{ijks})$.

Let R_{ijks} be the rank of X_{ijks} among all N observations and let $\overline{\mathbf{R}}_{k\cdot} = (\overline{R}_{11k\cdot}, \ldots, \overline{R}_{abk\cdot})'$ be the vector of the rank means for subject k and denote by $\widetilde{\mathbf{R}}_{\cdot\cdot} = n^{-1} \sum_{k=1}^n \overline{\mathbf{R}}_{k\cdot} = (\widetilde{R}_{11\cdot\cdot}, \ldots, \widetilde{R}_{ab\cdot\cdot})'$ the vector of the unweighted rank means over all subjects.

Let $\overline{\mathbf{Y}}_{k\cdot} = (\overline{Y}_{11k\cdot}, \ldots, \overline{Y}_{abk\cdot})'$ where $\overline{Y}_{ijk\cdot} = m_{ijk}^{-1} \sum_{s=1}^{m_{ijk}} Y_{ijks}$ and $Y_{ijks} = H(X_{ijks})$. Let $\widetilde{\mathbf{Y}}_{\cdot\cdot} = n^{-1} \sum_{k=1}^n \overline{\mathbf{R}}_{k\cdot}$ and let $\mathbf{C}$ be any suitable contrast matrix, then under $H_0 : \mathbf{CF} = \mathbf{0}$, the statistics $\sqrt{n}\mathbf{C} \int \widehat{H} d\widetilde{\mathbf{F}} = \sqrt{n}\mathbf{C}\widetilde{\mathbf{R}}_{\cdot\cdot}/N$ and $\sqrt{n}\mathbf{C} \int H d\widetilde{\mathbf{F}} = \sqrt{n}\mathbf{C}\widetilde{\mathbf{Y}}_{\cdot\cdot}$ are asymptotically equivalent which follows from Theorem 2.2. The covariance matrix of $\sqrt{n}\, \widetilde{\mathbf{Y}}_{\cdot\cdot}$ is denoted by $\mathbf{V}$ and it is assumed that $\mathbf{V}$ is nonsingular. A consistent estimate of $\mathbf{V}$, namely

$$\widehat{\mathbf{V}} = \frac{1}{N^2(n-1)} \sum_{k=1}^n \left(\overline{\mathbf{R}}_{k\cdot} - \widetilde{\mathbf{R}}_{\cdot\cdot}\right) \left(\overline{\mathbf{R}}_{k\cdot} - \widetilde{\mathbf{R}}_{\cdot\cdot}\right)' \quad (25)$$

follows from Theorem 2.3 and thus, $\mathbf{C}\, \widehat{\mathbf{V}}\, \mathbf{C}'$ is a consistent estimate of $Cov\left(\sqrt{n}\, \mathbf{C}\, \widetilde{\mathbf{Y}}_{\cdot\cdot}\right)$ where $\mathbf{C}$ is a suitable contrast matrix by which the hypothesis is formulated. Further notation will be explained along with the different test statistics.

Statistics

In general mixed models, where no compound symmetry is assumed, one can not expect that statistics can be given in terms of sums of squares like in the special ANOVA mixed models under the restrictive assumptions on the random effects.

Test for the Main Effect A

We note that the design is symmetric in A and B and therefore, only a test for the main effect A is derived. The test for B will follow by interchanging the indices i and j.

Hypothesis: $H_0^F(A): \left(\mathbf{P}_a \otimes \frac{1}{b}\mathbf{1}_b'\right)\mathbf{F} = \mathbf{0}$.
Notation:
$\mathbf{V}_a = Cov\left(\sqrt{n}\left(\mathbf{I}_a \otimes \frac{1}{b}\mathbf{1}_b'\right)\widetilde{\mathbf{Y}}_{\cdot\cdot}\right) = \left(\mathbf{I}_a \otimes \frac{1}{b}\mathbf{1}_b'\right)\mathbf{V}\left(\mathbf{I}_a \otimes \frac{1}{b}\mathbf{1}_b\right)$ is estimated by

$$\widehat{\mathbf{V}}_a = \left(\mathbf{I}_a \otimes \tfrac{1}{b}\mathbf{1}_b'\right)\ \widehat{\mathbf{V}}\ \left(\mathbf{I}_a \otimes \tfrac{1}{b}\mathbf{1}_b\right) = \frac{1}{N^2(n-1)}\sum_{k=1}^{n}\left(\widetilde{\mathbf{R}}_{k\cdot}^A - \widetilde{\mathbf{R}}_{\cdot\cdot}^A\right)\left(\widetilde{\mathbf{R}}_{k\cdot}^A - \widetilde{\mathbf{R}}_{\cdot\cdot}^A\right)'$$

where $\widetilde{\mathbf{R}}_{k\cdot}^A = (\widetilde{R}_{1\cdot k\cdot}, \ldots, \widetilde{R}_{a\cdot k\cdot})'$, $\widetilde{R}_{i\cdot k\cdot} = b^{-1}\sum_{j=1}^{b}\overline{R}_{ijk\cdot}$ and $\widetilde{\mathbf{R}}_{\cdot\cdot}^A = n^{-1}\sum_{k=1}^{n}\widetilde{\mathbf{R}}_{k\cdot}^A$. Let $\widehat{\mathbf{W}} = \widehat{\mathbf{V}}_a^{-1}\left(\mathbf{I}_a - \mathbf{J}_a\widehat{\mathbf{V}}_a^{-1}/\mathbf{1}_a'\widehat{\mathbf{V}}_a^{-1}\mathbf{1}_a\right)$ and denote the elements of $\widehat{\mathbf{V}}_a^{-1}$ by $\widehat{s}_{ii'}$, $i, i' = 1, \ldots, a$, and let $\widehat{s}_{i\cdot} = \sum_{i'=1}^{a}\widehat{s}_{ii'}$ and $\widehat{s}_{\cdot\cdot} = \sum_{i=1}^{a}\widehat{s}_{i\cdot}$.

Statistic:

$$\begin{aligned} Q_n(A) &= \sqrt{n}\left[(\mathbf{I}_a \otimes \tfrac{1}{b}\mathbf{1}_b')\,\widetilde{\mathbf{p}}\right]'\ \widehat{\mathbf{W}}\ \left[\left(\mathbf{I}_a \otimes \tfrac{1}{b}\mathbf{1}_b'\right)\widetilde{\mathbf{p}}\right]\sqrt{n} \\ &= \frac{n}{N^2}\left[\left(\widetilde{\mathbf{R}}_{\cdot\cdot}^A\right)'\widehat{\mathbf{V}}_a^{-1}\left(\widetilde{\mathbf{R}}_{\cdot\cdot}^A\right) - \left[\left(\widetilde{\mathbf{R}}_{\cdot\cdot}^A\right)'\widehat{\mathbf{V}}_a^{-1}\mathbf{1}_a\right]^2 / \mathbf{1}_a'\widehat{\mathbf{V}}_a^{-1}\mathbf{1}_a\right] \\ &= \frac{n}{N^2}\left[\sum_{i=1}^{a}\sum_{i'=1}^{a}\widetilde{R}_{i\cdots}\widehat{s}_{ii'}\widetilde{R}_{i'\cdots} - \frac{1}{\widehat{s}_{\cdots}}\left(\sum_{i=1}^{a}\widehat{s}_{i\cdot}\widetilde{R}_{i\cdots}\right)^2\right] \end{aligned} \tag{26}$$

has asymptotically ($n \to \infty$) a central χ_f^2-distribution with $f = a - 1$ d.f. under $H_0^F(A)$ and Q_n^A has the RT-property with respect to a parametric statistic for a two-way repeated measurements model with an unspecified structure of the covariance matrix.

Test for the Simple Factor A Effect

Hypothesis: $H_0^F(A|B): (\mathbf{P}_a \otimes \mathbf{I}_b)\mathbf{F} = \mathbf{0}$ or equivalently $H_0^F(A|B): (\mathbf{C}_A \otimes \mathbf{I}_b)\mathbf{F} = \mathbf{0}$ where $\mathbf{C}_A = (\mathbf{1}_{a-1} \vdots -\mathbf{I}_{a-1})$ is a $(a-1) \times a$ contrast matrix of full row rank.

Statistic:

$$Q_n(A|B) = \frac{n}{N^2}\widetilde{\mathbf{R}}_{\cdot\cdot}'\ (\mathbf{C}_A' \otimes \mathbf{I}_b)\ \left[(\mathbf{C}_A \otimes \mathbf{I}_b)\ \widehat{\mathbf{V}}\ (\mathbf{C}_A' \otimes \mathbf{I}_b)\right]^{-1}\ (\mathbf{C}_A \otimes \mathbf{I}_b)\,\widetilde{\mathbf{R}}_{\cdot\cdot} \tag{27}$$

has a central χ_f^2-distribution with $f = (a-1)b$ d.f. under $H_0^F(A|B)$ and $Q_n^{A|B}$ has the RT-property with respect to a parametric statistic for a two-way repeated measurements model with an unspecified structure of the covariance matrix.

Test for the Interaction AB

Hypothesis: $H_0^F(AB): (\mathbf{P}_a \otimes \mathbf{P}_b)\mathbf{F} = \mathbf{0}$ or equivalently $H_0^F(AB): (\mathbf{C}_A \otimes \mathbf{C}_B)\mathbf{F} = \mathbf{0}$ where $\mathbf{C}_A = (\mathbf{1}_{a-1} \vdots -\mathbf{I}_{a-1})$ and $\mathbf{C}_B = (\mathbf{1}_{b-1} \vdots -\mathbf{I}_{b-1})$ are both contrast matrices of full row rank.

Statistic:

$$Q_n(AB) = \frac{n}{N^2}\widetilde{\mathbf{R}}_{\cdot\cdot}'\ (\mathbf{C}_A' \otimes \mathbf{C}_B')\ \left[(\mathbf{C}_A \otimes \mathbf{C}_B)\ \widehat{\mathbf{V}}\ (\mathbf{C}_A' \otimes \mathbf{C}_B')\right]^{-1}\ (\mathbf{C}_A \otimes \mathbf{C}_B)\,\widetilde{\mathbf{R}}_{\cdot\cdot} \tag{28}$$

has a central χ_f^2-distribution with $f = (a-1)(b-1)$ d.f. under $H_0^F(AB)$ and Q_n^{AB} has the RT-property with respect to a parametric statistic for a two-way repeated measurements model with an unspecified structure of the covariance matrix.

Example 4.

We apply the procedures derived in this Section to the data given by Koch (1969) in Example 2, where repeated measurements on two crossed fixed factors for 8 pairs of animals are observed. Both fixed factors have two levels, namely 'ethionine' (E) and 'control' (C) for factor A (diet) and 'oxygen' (O) and 'nitrogen' (N) for factor B (gas). For the description of the trial and for the data set, we refer to Koch (1969), Example 2. Note that in this paper only the hypothesis of equality of all 4 treatment combinations (EO, EN, CO, CN) is considered. The statistics given in this Section in (26) and (28) however allow to consider separately the nonparametric hypotheses of no diet effect, of no gas effect and of no interaction. The ranks R_{ijk} of all 32 observations, the rank means for the four treatment combinations and the results are given in Table 3. Since the statistics Q_n^A, Q_n^B and Q_n^{AB} have the RT-property, they can be computed from the ranks by any suitable statistical software package. Note that the statistic Q_n^B is obtained by interchanging the indices i and j in Q_n^A which follows from the symmetry of the design. The p-values are obtained from the F-distribution with $f_1 = 1$ and $f_2 = n - 1 = 7$ d.f. which is used as a small sample approximation of the null distribution of the statistics.

Here insert Table 3.

3.4 Higher-Way Layouts

Here, we will give a brief outline on the application of the general results given in Section 2 to higher-way layouts. The results in Section 2 have been given in a general form which enables the extension to higher-way layouts in the same way as for the experimental designs in the theory of linear models. The hypotheses are formulated by means of the marginal distribution functions F_{ij}, $i = 1, \ldots, r$, $j = 1, \ldots, c$. The factor levels which are applied to a whole subject (whole-plot-factor) are numbered from $i = 1, \ldots, r$ and the factor levels which are applied only to a part of each subject (sub-plot-factor) are numbered from $j = 1, \ldots, c$. If there is only one whole-plot-factor with $i = 1, \ldots, a$ levels, factor A say, then $r = a$ and similarly, if there is only one sub-plot-factor with $j = 1, \ldots, b$ levels, factor B say, then $c = b$.

The case of two crossed sub-plot-factors, factor A with $i = 1, \ldots, a$ levels and factor B with $j = 1, \ldots, b$ levels and one whole-plot-factor with one level only (two factor block design) has been considered in Section 3.3.2. Here, $r = 1$ and $c = ab$ where the index j is split into two indices $j_1 = 1, \ldots, a$ and $j_2 = 1, \ldots, b$ which are renamed to $j_1 = i$ and $j_2 = j$. Below, it will be indicated how to apply the results of Section 2 to derive statistics for a split-plot-plot design, i.e. a design with three crossed factors A, B and C. The factors A $(i = 1, \ldots, a)$ and B $(j = 1, \ldots, b)$ are applied to the whole subjects which are nested under the AB-interaction while the levels of the factor C $(l = 1, \ldots, c)$ are applied within each subject (sub-plot-factor). The observations are written by independent vectors

$$\mathbf{X}_{ijk} = (X_{ij1k}, \ldots, X_{ijck})', \quad i = 1, \ldots, a, \ j = 1, \ldots, b, \ k = 1, \ldots, n_{ij}$$

where $X_{ijlk} \sim F_{ijl}$, $l = 1, \ldots, c$. To keep notation simple, only the case of one replication $m_{ijlk} = 1$ per factor level and subject combination is considered. Now, let $\mathbf{F} = (F_{111}, \ldots, F_{abc})'$ where the last index $l = 1, \ldots, c$ is running faster than the second index $j = 1, \ldots, b$ which is running faster than the first index $i = 1, \ldots, a$. Let $\mathbf{M}_A$ denote the contrast matrix for the levels of factor A, i.e. $\mathbf{M}_A = (\mathbf{1}_{a-1} \vdots - \mathbf{I}_{a-1})$ or $\mathbf{M}_A = \mathbf{P}_a = \mathbf{I}_a - \frac{1}{a}\mathbf{J}_a$ for testing the main effect of factor A, e.g. Let similarly, $\mathbf{M}_B = (\mathbf{1}_{b-1} \vdots - \mathbf{I}_{b-1})$ or $\mathbf{M}_B = \mathbf{P}_b = \mathbf{I}_b - \frac{1}{b}\mathbf{J}_b$ and let $\mathbf{M}_c$ be defined analogously for factor C. Then the nonparametric hypotheses in this design are:

$$
\begin{array}{rlcll}
H_0(A): & (\mathbf{M}_A \otimes b^{-1}\mathbf{1}_b' \otimes c^{-1}\mathbf{1}_c')\mathbf{F} & = & \mathbf{C}_A\mathbf{F} = \mathbf{0}, & \text{(main effect } A) \\
H_0(B): & (a^{-1}\mathbf{1}_a' \otimes \mathbf{M}_B \otimes c^{-1}\mathbf{1}_c')\mathbf{F} & = & \mathbf{C}_B\mathbf{F} = \mathbf{0}, & \text{(main effect } B) \\
H_0(C): & (a^{-1}\mathbf{1}_a' \otimes b^{-1}\mathbf{1}_b' \otimes \mathbf{M}_C)\mathbf{F} & = & \mathbf{C}_C\mathbf{F} = \mathbf{0}, & \text{(main effect } C) \\
H_0(AB): & (\mathbf{M}_A \otimes \mathbf{M}_B \otimes c^{-1}\mathbf{1}_c')\mathbf{F} & = & \mathbf{C}_{AB}\mathbf{F} = \mathbf{0}, & \text{(interaction } AB) \\
H_0(AC): & (\mathbf{M}_A \otimes b^{-1}\mathbf{1}_b' \otimes \mathbf{M}_C)\mathbf{F} & = & \mathbf{C}_{AC}\mathbf{F} = \mathbf{0}, & \text{(interaction } AC) \\
H_0(BC): & (a^{-1}\mathbf{1}_a' \otimes \mathbf{M}_B \otimes \mathbf{M}_C)\mathbf{F} & = & \mathbf{C}_{BC}\mathbf{F} = \mathbf{0}, & \text{(interaction } BC) \\
H_0(ABC): & (\mathbf{M}_A \otimes \mathbf{M}_B \otimes \mathbf{M}_C)\mathbf{F} & = & \mathbf{C}_{ABC}\mathbf{F} = \mathbf{0}, & \text{(interaction } ABC) \\
H_0(A|BC): & (\mathbf{M}_A \otimes \mathbf{I}_b \otimes \mathbf{I}_c)\mathbf{F} & = & \mathbf{C}_{A|BC}\mathbf{F} = \mathbf{0}, & \text{(simple factor } A \text{ effect)} \\
H_0(B|AC): & (\mathbf{I}_a \otimes \mathbf{M}_B \otimes \mathbf{I}_c)\mathbf{F} & = & \mathbf{C}_{B|AC}\mathbf{F} = \mathbf{0}, & \text{(simple factor } B \text{ effect)} \\
H_0(C|AB): & (\mathbf{I}_a \otimes \mathbf{I}_b \otimes \mathbf{M}_C)\mathbf{F} & = & \mathbf{C}_{C|AB}\mathbf{F} = \mathbf{0}, & \text{(simple factor } C \text{ effect).}
\end{array}
$$

Let $H = N^{-1}\sum_{i,j,l,k} F_{ijl} = N^{-1}\sum_{i,j,l} n_{ij}F_{ijl}$ where $N = c\sum_{i,j} n_{ij}$ and let $\widehat{H}$ and $\widehat{F}_{ijl}$ denote the empirical distribution functions. Let R_{ijlk} be the rank of X_{ijlk} and let $Y_{ijlk} = H(X_{ijlk})$ denote the ART of X_{ijlk}. Let

$$
\mathbf{V}_{ij} = Cov\left(\sqrt{N}\,\overline{\mathbf{Y}}_{ij\cdot}\right) = Cov\left(\sqrt{N}\left(\overline{Y}_{ij1\cdot}, \ldots, \overline{Y}_{ijc\cdot}\right)'\right)
$$

and since the vectors $\overline{\mathbf{Y}}_{ij\cdot}$ are independent by assumption,

$$
\begin{aligned}
\mathbf{V} &= Cov\left(\sqrt{N}\,\overline{\mathbf{Y}}_{\cdot}\right) = Cov\left(\sqrt{N}\left(\overline{Y}_{111\cdot}, \ldots, \overline{Y}_{abc\cdot}\right)'\right) \\
&= \bigoplus_{i=1}^{a}\bigoplus_{j=1}^{b} \mathbf{V}_{ij}
\end{aligned}
$$

and $\mathbf{V}_{ij}$ is estimated consistently by

$$
\widehat{\mathbf{V}}_{ij} = \frac{1}{Nn_{ij}(n_{ij}-1)}\sum_{k=1}^{n_{ij}}\left(\mathbf{R}_{ijk} - \overline{\mathbf{R}}_{ij\cdot}\right)\left(\mathbf{R}_{ijk} - \overline{\mathbf{R}}_{ij\cdot}\right)'
$$

where $\mathbf{R}_{ijk} = (R_{ij1k}, \ldots, R_{ijck})'$ and $\mathbf{R}_{ij\cdot} = n_{ij}^{-1}\sum_{k=1}^{n_{ij}} \mathbf{R}_{ijk}$.

The statistics for testing the nonparametric main effects A and C, e.g., can be derived by straightforward computation from Theorem 2.4 where $\mathbf{C}$ is taken as $\mathbf{C} = \mathbf{C}_A = (\mathbf{P}_a \otimes \frac{1}{b}\mathbf{1}_b' \otimes \frac{1}{c}\mathbf{1}_c')$ and $\mathbf{C} = \mathbf{C}_C = (\frac{1}{a}\mathbf{1}_a' \otimes \frac{1}{b}\mathbf{1}_b' \otimes \mathbf{P}_c)$, respectively.

Test for the Group Effect A

Hypothesis: $H_0^F(A) : (\mathbf{P}_a \otimes \frac{1}{b}\mathbf{1}_b' \otimes \frac{1}{c}\mathbf{1}_c')\mathbf{F} = \mathbf{0}$.
Notation:

$$\begin{aligned}
\tau_{ij}^2 &= N\sigma_{ij}^2 \ , \quad \sigma_{ij}^2 = Var(\overline{Y}_{ij\cdot\cdot}) \ , \quad \overline{Y}_{ij\cdot\cdot} = \frac{1}{n_{ij}}\sum_{k=1}^{n_{ij}} \overline{Y}_{ij\cdot k}, \\
\widehat{\tau}_{ij}^2 &= \frac{1}{Nn_{ij}(n_{ij}-1)}\sum_{k=1}^{n_{ij}}(\overline{R}_{ij\cdot k} - \overline{R}_{ij\cdot\cdot})^2 \ , \quad \overline{R}_{ij\cdot\cdot} = \frac{1}{n_{ij}}\sum_{k=1}^{n_{ij}} \overline{R}_{ij\cdot k} \\
\tau_i^2 &= \frac{1}{b^2}\sum_{j=1}^{b} \tau_{ij}^2 \ , \quad \widehat{\tau}_i^2 = \frac{1}{b^2}\sum_{j=1}^{b} \widehat{\tau}_{ij}^2 \ .
\end{aligned}$$

Let $\mathbf{V}_A = Cov\left(\sqrt{N}\left(\mathbf{I}_a \otimes \frac{1}{b}\mathbf{1}_b' \otimes \frac{1}{c}\mathbf{1}_c'\right)\overline{\mathbf{Y}}.\right) = diag\{\tau_1^2, \ldots, \tau_a^2\}$ and $\mathbf{W}_A = \mathbf{V}_A^{-1}(\mathbf{I}_a - \mathbf{J}_a\mathbf{V}_A^{-1}/\sum_{i=1}^{a}[1/\tau_i^2])$ and let $\widehat{\mathbf{W}}_a$ be the matrix corresponding to $\mathbf{W}_A$ where τ_i^2 is replaced by $\widehat{\tau}_i^2$.
Statistic:

$$\begin{aligned}
Q_N(A) &= \frac{1}{N}(\widetilde{R}_{1\cdots}, \ldots, \widetilde{R}_{a\cdots})\ \widehat{\mathbf{W}}_A\ (\widetilde{R}_{1\cdots}, \ldots, \widetilde{R}_{a\cdots})' \\
&= \sum_{i=1}^{a}\frac{1}{\widehat{\tau}_i^2}\left(\widetilde{R}_{i\cdots} - \frac{1}{\sum_{r=1}^{a}[1/\widehat{\tau}_r^2]}\ \sum_{r=1}^{a}\frac{\widetilde{R}_{r\cdots}}{\widehat{\tau}_r^2}\right)^2
\end{aligned}$$

has asymptotically a χ_f^2-distribution with $f = a-1$ d.f. and $Q_N(A)$ has the RT-property with respect to a parametric statistic for a split-plot-plot design with unspecified covariance matrices $\mathbf{V}_{ij}$ which may be also unequal.

Test for the Treatment Effect C

Hypothesis: $H_0^F(C) : \left(\frac{1}{a}\mathbf{1}_a' \otimes \frac{1}{b}\mathbf{1}_b' \otimes \mathbf{P}_c\right)\mathbf{F} = \mathbf{0}$.
Notation:

$$\begin{aligned}
\mathbf{V}_C &= Cov(\sqrt{N}\left(\tfrac{1}{a}\mathbf{1}_a' \otimes \tfrac{1}{b}\mathbf{1}_b' \otimes \mathbf{I}_c\right)\overline{\mathbf{Y}}.) = \frac{1}{a^2b^2}\sum_{i=1}^{a}\sum_{j=1}^{b}\mathbf{V}_{ij} \\
\widehat{\mathbf{V}}_C &= \frac{1}{a^2b^2}\sum_{i=1}^{a}\sum_{j=1}^{b}\widehat{\mathbf{V}}_{ij} \\
\mathbf{W}_C &= \mathbf{V}_C^{-1}(\mathbf{I}_c - \mathbf{J}_c\mathbf{V}_C^{-1}/\mathbf{1}_c'\mathbf{V}_C^{-1}\mathbf{1}_c) \ , \quad \widehat{\mathbf{W}}_C = \widehat{\mathbf{V}}_C^{-1}(\mathbf{I}_c - \mathbf{J}_c\widehat{\mathbf{V}}_C^{-1}/\widehat{s}_{\cdot\cdot})
\end{aligned}$$

where $\widehat{s}_{\cdot\cdot} = \sum_{l=1}^{c}\sum_{l'=1}^{c}\widehat{s}_{ll'}$ and $\widehat{s}_{ll'}$ is the (l,l')-element of $\widehat{\mathbf{V}}_C^{-1}$.
Statistic:

$$\begin{aligned}
Q_N(C) &= \frac{1}{N}\widetilde{\mathbf{R}}_{\cdots}'\ \widehat{\mathbf{W}}_C\ \widetilde{\mathbf{R}}_{\cdots} \\
&= \frac{1}{N}\left[\sum_{l=1}^{c}\sum_{l'=1}^{c}\widetilde{R}_{\cdot\cdot l\cdot}\ \widehat{s}_{ll'}\ \widetilde{R}_{\cdot\cdot l'\cdot} - \frac{1}{\widehat{s}_{\cdot\cdot}}\left(\sum_{l=1}^{c}\widehat{s}_{l\cdot}\widetilde{R}_{\cdot\cdot l\cdot}\right)^2\right]
\end{aligned}$$

has asymptotically a χ_f^2-distribution with $f = c - 1$ d.f. under $H_0^F(C)$ and $Q_N(C)$ has the RT-property with respect to a parametric statistic for a split-plot-plot design with unspecified covariance matrices $\mathbf{V}_{ij}$ which may be also unequal.

4 Small Sample Approximations and Simulation Results

In this Section, some simulations results for small samples in special designs shall be given. The following designs are analyzed:

1. Paired sample design with $b = 2$ treatment levels and $m_{jk} = 1$ replication (see Section 3.1).
2. Two-factor nested design with $a \geq 2$ treatment levels and $m_{ik} \geq 1$ replications per subject (see Section 3.2).
3. Two-factor cross-classified design with $b > 2$ treatment levels and $m_{jk} = 1$ replication (see Section 3.1).
4. Three-factor cross-classified design (two-factor block design) with $a = b = 2$ levels for both fixed factors and $m_{ijk} = 1$ replication (see Section 3.3.2).
5. Partially nested design (split-plot design) with $a = 2$ levels of the whole-plot factor A, $b \geq 2$ levels of the sub-plot factor B and $m_{ijk} = 1$ replication (see Section 3.3.1).

The following distributions have been used for the simulations: (1) rectangular, $R[0,1]$, (2) standard normal, $N(0,1)$, (3) exponential, $Ex(1)$, (4) log-normal, $exp(N(0,1))$. For each sample size and situation, $N = 5000$ simulations have been performed.

Results

1. Paired sample design

$X_{jk} = cA_k + \epsilon_{jk}$, $k = 1, \ldots, n$, $j = 1, 2$. The random variables A_k and ϵ_{jk} are independent and identically distributed according to the distributions (1) - (4), $c = 0, 1, 2$ and $n = 7, 10$.

Here insert Table 4.

2. Two-factor nested design

$X_{iks} = cA_{ik} + \epsilon_{iks}$, $i = 1, \ldots, a$, $k = 1, \ldots, n$, $s = 1, \ldots, m_{ik}$. The random variables A_{ik} and ϵ_{iks} are independent and identically distributed according to the distributions (1) - (4), $c = 0, 1, 2$, $a = 2, 4, 6, 10$, $n = 5, 10$ and $m_{ik} = 1, 3, 5$.

Here insert Table 5.

3. Two-factor cross-classified design

The covariance structure is generated by an additive model

$$X_{jk} = cA_k + \epsilon_{jk}\ ,\ k = 1, \dots, n\ , j = 1, \dots, b$$

like in the paired sample design. Simulations are performed for $n = 6, 10$ and $b = 4$, $n = 15, 20$ and $b = 10$ and for $c = 0, 1, 2$ where A_k and ϵ_{jk} are distributed as in the paired sample design. The small sample approximation is motivated by the distribution of Hotelling's T^2 under the assumption of multivariate normality where the hypothesis $H_0^\mu : \mu_1 = \cdots = \mu_b$ is tested. Thus, $(n-b+1)Q_n^M(B)/[(b-1)(n-1)] \sim F_{f_1,f_2}$ where $f_1 = b - 1$ and $f_2 = n - b + 1$. The simulations have been performed for the distributions (1) - (4).

Here insert Table 6 and Table 7.

4. Three-factor cross-classified design $(2 \times n)$

Here, the covariance structure is generated as in the previous paragraph. Moreover, the null distributions of the statistics for the main effect A and for the interaction AB are simulated also when there is an additive effect of factor B. The case of an additive effect of one factor and of the interaction is also simulated. Let $\mu_{ij} = E(X_{ijk})$, $k = 1, \dots, n$, $i, j = 1, 2$, then four cases are considered:

	μ_{11}	μ_{12}	μ_{21}	μ_{22}	
(1)	0	0	0	0	no effect
(2)	1	0	1	0	main effect B
(3)	1	0	0	1	interaction
(4)	2	0	1	1	main effect B and interaction

The additive model is: $X_{ijk} = \mu_{ij} + cA_k + \epsilon_{ijk}$ where A_k and ϵ_{ijk} are distributed as described in the previous paragraph. The null distribution of $Q_n(A)$ given in (26) is simulated in the cases (2), (3) and (4). In case (1), the null distribution of $Q_n(A)$ and $Q_n(AB)$ given in (28) is simulated. Since there are only two levels for each fixed factor, the statistics are univariate statistics and the F-distribution with $f_1 = 1$ and $f_2 = n - 1$ d.f. is used as a small sample approximation. Simulations are performed for $n = 7, 10$ and $c = 0, 1, 2$.

Here insert Table 8.

5. Partially nested design $2n(2) \times b$

Only the case of $a = 2$ groups has been considered. The covariance structure is generated by an additive model

$$X_{ijk} = \beta_j + (\alpha\beta)_{ij} + cA_{ik} + \epsilon_{ijk}\ ,\ i = 1, 2\ , j = 1, \dots, b\ , k = 1, \dots, n_i \equiv n$$

like in the paired sample design. Simulations are performed for $b = 4$ and $n_1 = n_2 = n = 7, 10$ and for $b = 10$ and $n_1 = n_2 = n = 15, 20$ where $c = 0, 1, 2$ and A_k and ϵ_{jk} are independent and identically distributed according to the distributions (1) - (4). The small sample approximation is motivated by the distribution of the two-sample MANOVA-statistic under the assumption of multivariate normality (with equal covariance matrices) where the hypotheses $H_0(B) : \beta_1 = \cdots = \beta_b$ and $H_0(AB) : (\alpha\beta)_{11} - (\alpha\beta)_{21} = \cdots = (\alpha\beta)_{1b} - (\alpha\beta)_{2b}$ are tested. The approximation may be not so satisfactory as in the two-factor mixed model since in the nonparametric model, the covariance matrices are unequal in general. However, the approximation turned out to be rather good when $F_B = (2n-b)Q_n(B)/[(b-1)(2n-2)]$ and $F_{AB} = (2n-b)Q_n(AB)/[(b-1)(2n-2)]$ were compared with the F_{f_1,f_2}-distribution with $f_1 = b-1$ and $f_2 = 2n-b$ d.f. where $Q_n(B)$ and $Q_n(AB)$ are given in (19) and (21), respectively. The null distribution of F_B was also considered in the presence of the interaction $\beta_j = (j-1)/b$ for $i = 1$ and $\beta_j = (1-j)/b$ for $i = 2$. Similarly, the null distribution of F_{AB} was also considered in the presence of the main effect B, $\beta_j = (j-1)/b$ for $i = 1, 2$. The approximation by the limiting χ^2_{b-1}-distribution turned out to be very poor. The results are displayed in Tables 9 and 10.

Here insert Table 9 and Table 10.

In all simulations, the approximation by the asymptotic χ^2-distribution turned out to be liberal. If more parameters of the covariance matrix had to be estimated the approximation became worse - as was to be expected. The approximation by the null distribution of the respective statistic under normality assumption however, turned out to be extremely good even when the sample sizes are rather small. It may also be noted that the simulation results were almost the same for all distribution functions used in our study. Thus, it can be recommended to use the small sample approximations that have been examined here. The question is however, which approximation should be used in other designs than analyzed in this study.

Primarily, this question is not a problem of the nonparametric theory rather than a problem which is also open in many cases in parametric mixed models. In recent years, many authors have considered this problem in parametric models under the assumption of multivariate normal distributions, see e.g. Huynh & Feldt (1979), McLean & Sanders (1988), Schluchter & Elashoff (1990), Fai & Cornelius (1993) and references cited therein. The p-values produced by the different approximations suggested in the afore mentioned papers may differ considerably. Nevertheless, these approximations are available in the new SAS-procedure 'PROC MIXED' as options in the MODEL-statement and in the SAS/STAT manual (Changes and Enhancements, Release 6.10, 1994) it is stated: "The Satterthwaite method implemented here is intended to produce an accurate F-approximation; however, the results may differ from those produced by PROC GLM. Also, the small-sample properties of this approximation have not been extensively investigated for the various models available with PROC MIXED." This can really not be considered as a satisfactory solution of the problem, not even for parametric models. It is apparent that future research is necessary in this area.

Appendix

A Matrix Algebra and Handling of Ties

Lemma A.1 *Let* $\mathbf{V}_a$ *be any non-singular symmetric* $(a \times a)$ *matrix and let*

$$\mathbf{W}_a = \mathbf{V}_a^{-1}\left[\mathbf{I}_a - \mathbf{J}_a\mathbf{V}_a^{-1}/\mathbf{1}_a'\mathbf{V}_a^{-1}\mathbf{1}_a\right] .$$

Then, $\mathbf{W}_a\mathbf{1}_a = \mathbf{0}$ *and* $\mathbf{V}_a$ *is a generalized inverse of* $\mathbf{W}_a$*, i.e.* $\mathbf{W}_a\mathbf{V}_a\mathbf{W}_a = \mathbf{W}_a$.

Proof: Both statements follow easily by observing that

$$\mathbf{J}_a\mathbf{V}_a^{-1}\mathbf{1}_a/\mathbf{1}_a'\mathbf{V}_a^{-1}\mathbf{1}_a = \mathbf{1}_a\mathbf{1}_a'\mathbf{V}_a^{-1}\mathbf{1}_a/\mathbf{1}_a'\mathbf{V}_a^{-1}\mathbf{1}_a = \mathbf{1}_a .$$

Note that $\mathbf{1}_a'\mathbf{V}_a^{-1}\mathbf{1}_a = trace(\mathbf{V}_a^{-1})$ if $\mathbf{V}_a$ is diagonal. □

Next, we show that the hypotheses $H_0^F : \mathbf{W}_a\mathbf{F} = \mathbf{0}$ and $H_0^F : \mathbf{P}_a\mathbf{F} = \mathbf{0}$ are identical. It suffices to show that the solution spaces of the homogeneous linear equations systems are identical.

Lemma A.2 *Let* $\mathbf{W}_a$ *and* $\mathbf{V}_a$ *as in Lemma A.1 and let* $\mathbf{x} \in I\!R^a$*. Then the solution spaces of* $\mathbf{W}_a\mathbf{x} = \mathbf{0}$ *and* $\mathbf{P}_a\mathbf{x} = \mathbf{0}$ *are identical.*

Proof: The solution spaces are given by $\mathbf{x}_1 = (\mathbf{I}_a - \mathbf{W}_a^-\mathbf{W}_a)\mathbf{z}$ and $\mathbf{x}_2 = (\mathbf{I}_a - \mathbf{P}_a^-\mathbf{P}_a)\mathbf{z}$ where $\mathbf{z}$ is an arbitrary vector and $\mathbf{W}_a^-$ denotes a generalized inverse of $\mathbf{W}_a$ and $\mathbf{P}_a^-$ a generalized inverse of $\mathbf{P}_a$. Note that $\mathbf{V}_a$ is a generalized inverse of $\mathbf{W}_a$ and $\mathbf{P}_a^- = \mathbf{P}_a$ since $\mathbf{P}_a$ is a projection matrix. Thus, $\mathbf{I}_a - \mathbf{W}_a^-\mathbf{W}_a = \mathbf{I}_a - [\mathbf{I}_a - \mathbf{J}_a\mathbf{V}_a^{-1}/\mathbf{1}_a'\mathbf{V}_a^{-1}\mathbf{1}_a] = \mathbf{J}_a\mathbf{V}_a^{-1}/\mathbf{1}_a'\mathbf{V}_a^{-1}\mathbf{1}_a$ and $\mathbf{I}_a - [\mathbf{I}_a - \frac{1}{a}\mathbf{J}_a] = \frac{1}{a}\mathbf{J}_a$. Then $\mathbf{x}_1 = \mathbf{1}_a z_1$ and $\mathbf{x}_2 = \mathbf{1}_a z_2$ where $z_1 = \mathbf{1}_a'\mathbf{V}_a^{-1}\mathbf{z}/\mathbf{1}_a'\mathbf{V}_a^{-1}\mathbf{1}_a$ and $z_2 = \mathbf{1}_a'\mathbf{z}/a$ are arbitrary constants which proves the result. □

Below we give the simple results which are needed for the handling of ties.

Lemma A.3 *Let* $X_{ij} \sim F_i$*,* $i = 1, \ldots, c$*,* $j = 1, \ldots, n_i$ *independent random variables and let* F_i *as given in (1) and let* $c(u) = [c^+(u) + c^-(u)]/2$ *as defined in connection with (2). Then*

1. $E\left[c(x - X_{ij}) - F_i(x)\right] = 0$,
2. $E\left[c(X_{ij} - X_{is}) - F_i(X_{ij})\right] = 0$, *if* $s \neq j$,
3. $\int F_i dF_i = \frac{1}{2}$.

Proof: (1) By definition, $E\left[c(x - X_{ij}) - F_i(x)\right] = P(X_{ij} < x) + \frac{1}{2}P(X_{ij} = x) - F_i(x) = F_i^-(x) + \frac{1}{2}\left[F_i^+(x) - F_i^-(x)\right] - F_i(x) = \frac{1}{2}\left[F_i^+(x) + F_i^-(x)\right] - F_i(x) = 0.$
(2) follows by noting that $E\left(\left[c(X_{ij} - X_{is}) - F_i(X_{ij})\right] | X_{ij} = x\right) = 0$, if $s \neq j$.
(3) The result follows using integration by parts (see Hewitt & Stromberg (1969), p. 419).
□

B Moment Inequalities

To prove the asymptotic results, we give first some moment inequalities for empirical processes which are needed in several places.

Consider the notation in (6). For convenience, the two indices i, k, $i = 1, \ldots, r$ and $k = 1, \ldots, n_i$ are collapsed to one index $i' = 1, \ldots, n$ where $n = \sum_{i=1}^{r} n_i$. Also the two indices $j = 1, \ldots, c$ and $s = 1, \ldots, m_{ij}$ are collapsed to one index $j' = 1, \ldots, M_i$ where $M_i = \sum_{j=1}^{c} m_{ij}$. For simplicity, however, we will use i, j instead of i', j'. Note that with this notation, the random variables X_{ij} and X_{rs} are independent if $i \neq r$ and may be dependent if $i = r$. Moreover, we generalize the assumptions of the general mixed model (6) and assume that $X_{ij} \sim F_{ij}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, M_i$ where $|M_i| \leq M < \infty$. Since the random variables X_{ij} and X_{is} may be dependent, we will use the Cramer-Wold device to state asymptotic normality.

Let λ_{ij} be suitable weights with $|\lambda_{ij}| \leq \lambda_M < \infty$ and let $N = \sum_{i=1}^{n} M_i$. Then we define

$$\begin{aligned} H(x) &= \frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{M_i} F_{ij}(x) \quad , \quad \widehat{H}(x) = \frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{M_i} c(x - X_{ij}) \\ G_N(x) &= \frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{M_i} \lambda_{ij} F_{ij}(x)\ , \quad \widehat{G}_N(x) = \frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{M_i} \lambda_{ij} c(x - X_{ij})\ . \end{aligned} \tag{29}$$

Finally we denote by

$$p_N = \int H dG_N = \frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{M_i} \lambda_{ij} \int H dF_{ij}$$

a linear combination of the generalized means $p_{ij} = \int H dF_{ij}$.

Definition B.1 *Let R_{ij} be the rank of X_{ij} among all N observations. Then the statistic*

$$\widehat{p}_N = \int \widehat{H} d\widehat{G}_N = \frac{1}{N^2}\sum_{i=1}^{n}\sum_{j=1}^{M_i} \lambda_{ij}\left(R_{ij} - \tfrac{1}{2}\right)$$

is called a 'linear rank statistic' in the mixed model.

Next we state the basic moment inequalities.

Lemma B.2 *With the notation introduced in (29), we have*

$$\begin{aligned} (1)\quad & E\left[\widehat{G}_N(x) - G_N(x)\right]^2 \leq \frac{\lambda_M^2}{N^2}\sum_{k=1}^{n} M_k^2 \\ (2)\quad & E\left[\widehat{G}_N(X_{ij}) - G_N(X_{ij})\right]^2 \leq \frac{\lambda_M^2}{N^2}\sum_{k=1}^{n} M_k^2\ . \end{aligned}$$

Proof: To prove (1), note that

$$\begin{aligned} & E\left[\widehat{G}_N(x) - G_N(x)\right]^2 \\ &= \frac{1}{N^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{M_i}\sum_{s=1}^{M_k}\lambda_{ij}\lambda_{ks}E\left([c(x-X_{ij})-F_{ij}(x)]\,[c(x-X_{ks})-F_{ks}(x)]\right) \\ &= \frac{1}{N^2}\sum_{i=1}^{n}E\left(\sum_{j=1}^{M_i}\lambda_{ij}\,[c(x-X_{ij})-F_{ij}(x)]\right)^2 \\ &\leq \frac{1}{N^2}\sum_{i=1}^{n}E\left(M_i\sum_{j=1}^{M_i}\lambda_{ij}^2\,[c(x-X_{ij})-F_{ij}(x)]^2\right) \\ &\leq \frac{\lambda_M^2}{N^2}\sum_{i=1}^{n}M_i^2 \end{aligned}$$

by independence of X_{ij} and X_{ks} for $i \neq k$ and using Jensen's inequality. Statement (2) follows in the same way by noting that

$$E\left([c(X_{ij}-X_{ar})-F_{ar}(X_{ij})]\,[c(X_{ij}-X_{bs})-F_{bs}(X_{ij})]\right) = 0$$

if $a \neq b$ since either $i \neq a$ or $i \neq b$ in this case. □

C Asymptotic Results

Proof of Lemma 2.1

We first restate Lemma 2.1 according to the condensed (and more general) notation introduced in Appendix B.

Lemma C.1 *Let $\mathbf{X}_i = (X_{i1}, \ldots, X_{iM_i})'$, $i = 1, \ldots, n$, be independent random vectors where $X_{ij} \sim F_{ij}$, $j = 1, \ldots, M_i \leq M < \infty$. Then the linear rank statistic $\widehat{p}_N$ given in Definition B.1 is 'consistent' for $p_N = \int H dG_N$ in the sense that $\widehat{p}_N - p_N \xrightarrow{p} 0$ as $n \to \infty$.*

Proof: We will show that $E\left(\int \widehat{H} d\widehat{G}_N - \int H dG_N\right)^2 \to 0$. By the same arguments as in the proof of Lemma B.2, (4) it follows

$$\begin{aligned} E(\widehat{p}_N - p_N)^2 &= E\left[\int \widehat{H} d\widehat{G}_N - \int H dG_N\right]^2 \\ &\leq \frac{2\lambda_M^2}{N}\sum_{i=1}^{n}\sum_{j=1}^{M_i}E\left(\widehat{H}(X_{ij}) - H(X_{ij})\right)^2 + 2\int E(\widehat{G}_N - G_N)^2 dH \\ &\leq \frac{4M\lambda_M^2}{N} = O\left(\frac{1}{N}\right) . \end{aligned}$$

□

The statement of Lemma 2.1 follows by a suitable choice of the weights λ_{ij} for the linear rank statistic $\widehat{p}_N$.

Proof of Theorem 2.2

To prove Theorem 2.2, we state the asymptotic equivalence of a linear rank statistic and a statistic which can be written as a sum of independent random variables. Theorem 2.2 follows from this more general result by a suitable choice of the weights λ_{ij} for the linear rank statistic $\widehat{p}_N$.

Theorem C.2 *Let $\mathbf{X}_i$ be as in Lemma C.1 and assume that $M_i \le M < \infty$. Then*

1. $\sqrt{N}\left[\int \widehat{H} d\widehat{G}_N - \int H dG_N\right] \doteq \sqrt{N}\left[\int H d\widehat{G}_N + \int \widehat{H} dG_N - 2\int H dG_N\right] = \sqrt{N} B_N,$

2. $\sqrt{N}\int \widehat{H} d\left(\widehat{G}_N - G_N\right) \doteq \sqrt{N}\int H d\left(\widehat{G}_N - G_N\right) = \sqrt{N} D_N$

as $n \to \infty$.

Proof: First, we decompose

$$\int \widehat{H} d\widehat{G}_N - \int H dG_N \;=\; \int H d\widehat{G}_N + \int \widehat{H} dG_N - 2\int H dG_N + \int (\widehat{H} - H) d(\widehat{G}_N - G_N)\,.$$

It will be shown that $E\left(\sqrt{N}\int (\widehat{H} - H) d(\widehat{G}_N - G_N)\right)^2 \to 0.$

$$\begin{aligned}
&\left(\sqrt{N}\int (\widehat{H} - H) d(\widehat{G}_N - G_N)\right)^2 \\
&= N\left(\frac{1}{N}\sum_{a=1}^{n}\sum_{s=1}^{M_a}\lambda_{as}\left[\widehat{H}(X_{as}) - H(X_{as}) - \int\left(\widehat{H}(x) - H(x)\right) dF_{as}(x)\right]\right)^2 \\
&= N\left(\frac{1}{N^2}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{s=1}^{M_a}\sum_{t=1}^{M_b}\lambda_{as}\left[c(X_{as} - X_{bt}) - F_{bt}(X_{as}) - \int (c(x - X_{bt}) - F_{bt}(x))\, dF_{as}(x)\right]\right)^2 \\
&= \frac{1}{N^3}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{c=1}^{n}\sum_{d=1}^{n}\sum_{s=1}^{M_a}\sum_{t=1}^{M_b}\sum_{u=1}^{M_c}\sum_{v=1}^{M_d}\lambda_{as}\lambda_{cu}\varphi(X_{as}, X_{bt})\varphi(X_{cu}, X_{dv})
\end{aligned}$$

where $\varphi(X_{as}, X_{bt}) = c(X_{as} - X_{bt}) - F_{bt}(X_{as}) - \int (c(x - X_{bt}) - F_{bt}(x))\, dF_{as}(x)$. If one of the indices a, b, c or d is different from all three others, then $E[\varphi(X_{as}, X_{bt})\varphi(X_{cu}, X_{dv})] = 0$ since it follows by independence and from Lemma A.3 that $E\left(\varphi(X_{as}, X_{bt}) | X_{as} = x\right) = 0$ and $E\left(\varphi(X_{as}, X_{bt}) | X_{bt} = x\right) = 0$. Thus,

$$\begin{aligned}
&E\left(\sqrt{N}\int (\widehat{H} - H) d(\widehat{G}_N - G_N)\right)^2 \\
&\quad\ll \frac{1}{N^3}\sum_{a=1}^{n}\sum_{c=1}^{n}\sum_{s=1}^{M_a}\sum_{t=1}^{M_a}\sum_{u=1}^{M_c}\sum_{v=1}^{M_c}|\lambda_{as}\lambda_{cu}| \\
&\quad\ll \frac{\lambda_M^2}{N^3}\left(\sum_{a=1}^{n} M_a^2\right)^2 \ll \frac{\lambda_M^2 M^2}{N} = o(1) \quad \text{if } n \to \infty
\end{aligned}$$

where the Vinogradov symbol '$\ll$' is used instead of the $O(\cdot)$-notation, for simplicity. Next, note that the relation in statement (2) follows by subtracting $\sqrt{N}\int(\widehat{H}-H)dG_N$ from both sides of the relation (1). Thus, statement (2) follows from (1). □

Remark: The statement of Theorem 2.2 follows by a suitable choice of the weights λ_{ij} for $\widehat{p}_N$ and using the condition $\min n_i \to \infty$.

The asymptotic normality of $\sqrt{N}\widehat{p}_N$ under the hypotheses $H_0^F : G_N = 0$ and $H_0 : \int HdG_N = 0$ is stated in the next Theorem.

Theorem C.3 *Let B_N and D_N as in Theorem C.2 and let $\sigma_N^2 = Var\,(NB_N)$ and $\tau_N^2 = Var\,(ND_N)$. If $\sigma_N^2 \to \infty$ as $n \to \infty$, then*

1. *under $H_0 : G_N = 0$,*
$$\frac{N}{\tau_N}\int \widehat{H}d\widehat{G}_N \xrightarrow{\mathcal{L}} U \sim N(0,1)\ ,$$
2. *under $H_0 : \int HdG_N = 0$,*
$$\frac{N}{\sigma_N}\int \widehat{H}d\widehat{G}_N \xrightarrow{\mathcal{L}} U \sim N(0,1)\ .$$

Proof: (1) Under $H_0 : G_N = 0$, it follows from Theorem C.2 that $B_N = D_N$ and $\sigma_N^2 = \tau_N^2$. The result follows by the Lindeberg-Feller Theorem since $H(\cdot)$ is uniformly bounded and $\sigma_N^2 \to \infty$, by assumption. The result for (2) follows directly from Theorem C.2, (1) and by the Lindeberg-Feller Theorem. For details see Brunner & Denker (1994). □

Proof of Theorem 2.3

Next, the consistency of the estimator for the covariance matrix will be shown.

Let $\widetilde{Y}_{ij\cdot\cdot} = \int Hd\widetilde{F}_{ij}$, $\widehat{\widetilde{Y}}_{ij\cdot\cdot} = \int \widehat{H}d\widetilde{F}_{ij}$, $\overline{Y}_{ijk\cdot} = \int Hd\widehat{F}_{ijk}$ and $\widehat{\overline{Y}}_{ijk\cdot} = \int \widehat{H}d\widehat{F}_{ijk}$ where $\widehat{F}_{ijk}(x) = m_{ijk}^{-1}\sum_{s=1}^{m_{ijk}} c(x - X_{ijks})$ and $\widetilde{F}_{ij} = n_i^{-1}\sum_{k=1}^{n_i}\widehat{F}_{ijk}$.

We want to estimate

$$\sigma_{ij}^2 \;=\; Var\left(\sqrt{N}\,\widetilde{Y}_{ij\cdot\cdot}\right) = \frac{1}{n_i}\sum_{k=1}^{n_i}\frac{N}{n_i}s_{ijk}^2$$

where $s_{ijk}^2 = Var(\overline{Y}_{ijk\cdot}) = Var\left(m_{ijk}^{-1}\sum_{s=1}^{m_{ijk}} H(X_{ijks})\right)$. First, we define an 'estimator' with unobservable random variables.

$$\widetilde{\sigma}_{ij}^2 \;=\; \frac{N}{n_i(n_i-1)}\sum_{k=1}^{n_i}\left(\overline{Y}_{ijk\cdot} - \widetilde{Y}_{ij\cdot\cdot}\right)^2$$

and it will be shown that $\widetilde{\sigma}_{ij}^2 - \sigma_{ij}^2 \xrightarrow{p} 0$. Note that the random variables $\overline{Y}_{ijk\cdot}$, $k = 1, \ldots, n_i$ are independent and that $|\overline{Y}_{ijk\cdot}| \leq 1$. Let $Z_{ijk} = (\overline{Y}_{ijk\cdot} - \mu_{ij})^2$ where $\mu_{ij} = E(\overline{Y}_{ijk\cdot})$, $k = 1, \ldots, n_i$. Note that $s_{ijk}^2 = E(Z_{ijk})$. Then

$$\begin{aligned}
\widetilde{\sigma}_{ij}^2 - \sigma_{ij}^2 &= \frac{N}{n_i(n_i-1)} \sum_{k=1}^{n_i} \left(\overline{Y}_{ijk\cdot} - \widetilde{Y}_{ij\cdot\cdot}\right)^2 - \frac{N}{n_i^2} \sum_{k=1}^{n_i} s_{ijk}^2 \\
&= \frac{N}{n_i^2} \sum_{k=1}^{n_i} \left[\left(\overline{Y}_{ijk\cdot} - \mu_{ij} - \left(\widetilde{Y}_{ij\cdot\cdot} - \mu_{ij}\right)\right)^2 - s_{ijk}^2\right] + \frac{N}{n_i^2(n_i-1)} \sum_{k=1}^{n_i} \left(\overline{Y}_{ijk\cdot} - \widetilde{Y}_{ij\cdot\cdot}\right)^2 \\
&= \frac{N}{n_i^2} \sum_{k=1}^{n_i} (Z_{ijk} - E(Z_{ijk})) - \frac{N}{n_i} \left(\widetilde{Y}_{ij\cdot\cdot} - \mu_{ij}\right)^2 + \frac{N}{n_i^2(n_i-1)} \sum_{k=1}^{n_i} \left(\overline{Y}_{ijk\cdot} - \widetilde{Y}_{ij\cdot\cdot}\right)^2 \\
&= A_{ij} - B_{ij} + C_{ij} \; .
\end{aligned}$$

We consider the three terms separately and note that N/n_i is uniformly bounded by assumption (Theorem 2.2) and $s_{ijk}^2 = E(Z_{ijk}) \leq 1$ and $Var(Z_{ijk}) \leq 1$ since $|Z_{ijk}| \leq 1$.

By the strong law of large numbers it follows that $A_{ij} \xrightarrow{a.s.} 0$ if $\min n_i \to \infty$ since Kolmogorov's condition $\sum_{k=1}^{\infty} k^{-2} Var(Z_{ijk}) < \infty$ follows from $Var(Z_{ijk}) \leq 1$.

For the second term, we note that Kolmogorov's condition $\sum_{k=1}^{\infty} k^{-2} s_{ijk}^2 < \infty$ follows from $s_{ijk}^2 \leq 1$ and thus, by the strong law of large numbers,

$$\frac{N}{n_i} \left(\widetilde{Y}_{ij\cdot\cdot} - \mu_{ij}\right)^2 = \frac{N}{n_i} \left(\widetilde{Y}_{ij\cdot\cdot} - \mu_{ij}\right) \frac{1}{n_i} \sum_{k=1}^{n_i} \left(\overline{Y}_{ijk\cdot} - \mu_{ij}\right) \xrightarrow{a.s.} 0$$

if $\min n_i \to \infty$ since $|\widetilde{Y}_{ij\cdot\cdot} - \mu_{ij}| \leq 1$. For the last term, note that $|\overline{Y}_{ijk\cdot} - \widetilde{Y}_{ij\cdot\cdot}| \leq 1$ and thus, $C_{ij} \leq N/[n_i(n_i-1)] \to 0$.

Putting everything together, it follows that $\widetilde{\sigma}_{ij}^2 - \sigma_{ij}^2 \xrightarrow{p} 0$.

Next, the unobservable random variables $Y_{ijks} = H(X_{ijks})$ are replaced by observable random variables $\widehat{Y}_{ijks} = \widehat{H}(X_{ijks}) = N^{-1}(R_{ijks} - 1/2)$ where R_{ijks} is the rank of X_{ijks} among all N random variables. To this end, define

$$\begin{aligned}
\widehat{\sigma}_{ij}^2 &= \frac{1}{Nn_i(n_i-1)} \sum_{k=1}^{n_i} \left(\overline{R}_{ijk\cdot} - \widetilde{R}_{ij\cdot\cdot}\right)^2 = \frac{N}{n_i(n_i-1)} \sum_{k=1}^{n_i} \left(\widehat{\overline{Y}}_{ijk\cdot} - \widehat{\widetilde{Y}}_{ij\cdot\cdot}\right)^2 \\
&= \frac{N}{n_i(n_i-1)} \left[\sum_{k=1}^{n_i} \widehat{\overline{Y}}_{ijk\cdot}^2 - n_i \widehat{\widetilde{Y}}_{ij\cdot\cdot}^2\right] .
\end{aligned}$$

Then it will be shown that $E\left(\widehat{\sigma}_{ij}^2 - \widetilde{\sigma}_{ij}^2\right)^2 \to 0$.

$$\begin{aligned}
&\left(\widehat{\sigma}_{ij}^2 - \widetilde{\sigma}_{ij}^2\right)^2 \\
&\quad = \frac{N^2}{n_i^2(n_i-1)^2} \left[\sum_{k=1}^{n_i} \left(\widehat{\overline{Y}}_{ijk\cdot}^2 - \overline{Y}_{ijk\cdot}^2\right) - n_i \left(\widehat{\widetilde{Y}}_{ij\cdot\cdot}^2 - \widetilde{Y}_{ij\cdot\cdot}^2\right)\right]^2 \\
&\quad = \frac{N^2}{n_i^2(n_i-1)^2} \left[\sum_{k=1}^{n_i} \left(\widehat{\overline{Y}}_{ijk\cdot} - \overline{Y}_{ijk\cdot}\right)\left(\widehat{\overline{Y}}_{ijk\cdot} + \overline{Y}_{ijk\cdot}\right) - n_i \left(\widehat{\widetilde{Y}}_{ij\cdot\cdot} - \widetilde{Y}_{ij\cdot\cdot}\right)\left(\widehat{\widetilde{Y}}_{ij\cdot\cdot} + \widetilde{Y}_{ij\cdot\cdot}\right)\right]^2
\end{aligned}$$

$$\leq \frac{8N^2}{n_i^2(n_i-1)^2}\left[n_i\sum_{k=1}^{n_i}\left(\widehat{Y}_{ijk\cdot}-\overline{Y}_{ijk\cdot}\right)^2+n_i^2\left(\int(\widehat{H}-H)d\widetilde{F}_{ij}\right)^2\right]$$
$$\leq \frac{8N^2}{n_i^2(n_i-1)^2}\left[n_i^2\int(\widehat{H}-H)^2d\widetilde{F}_{ij}+n_i^2\int(\widehat{H}-H)^2d\widetilde{F}_{ij}\right]$$

using Jensen's inequality. Finally, it follows

$$E\left(\widehat{\sigma}_{ij}^2-\widetilde{\sigma}_{ij}^2\right)^2 \leq \frac{16N^2}{(n_i-1)^2}E\left(\int(\widehat{H}-H)^2d\widetilde{F}_{ij}\right)\leq\frac{16c^2M^2}{(n_i-1)^2}n_i=O\left(\frac{1}{n_i}\right)$$

by Lemma B.2. The covariances are estimated in the same way and the result follows. □

Outline of the Proof of Theorem 2.4

Statement (1) follows from Theorem 2.2 and Theorem C.3 by the Cramer-Wold device for a suitable choice of the weights λ_{ij} for $\widehat{p}_N$ and using the assumptions $A1$ and $A3$. Note that the condition $\sigma_N^2 \to \infty$ in Theorem C.3 is satisfied since the $\widetilde{\mathbf{Y}}_{i\cdot\cdot}$ are independent and $Var(N\mathbf{d}_i'\widetilde{\mathbf{Y}}_{i\cdot\cdot}) = N\mathbf{d}_i'\mathbf{V}_i\mathbf{d}_i \geq N\|\mathbf{d}_i\|\lambda_{min} \to \infty$ for any fixed vector $\mathbf{d}_i$ with $\|\mathbf{d}_i\| \neq 0$ where $\lambda_{min} \geq k_0' > 0$ is the smallest eigenvalue of $\mathbf{V}_i$. Note also that $\lambda_{min} \geq k_0' > 0$ is equivalent to the assumption $|\mathbf{V}_i| \geq k_0 > 0$.

To prove (2), note that $\mathbf{X}'\mathbf{S}^-\mathbf{X} \sim \chi_f^2$ with $f = rank(\mathbf{S})$ if $\mathbf{S}^-$ is a symmetric reflexive generalized inverse of $\mathbf{S}$ and if $\mathbf{X} \sim N(\mathbf{0}, \mathbf{S})$, see e.g. Rao & Mitra (1971), Theorem 9.2.3. Let $\mathbf{X} = \sqrt{N}\mathbf{C}\widetilde{\mathbf{p}}$, then under H_0^F, $\mathbf{X}$ has asymptotically a multivariate normal distribution with mean $\mathbf{0}$ and covariance matrix $\mathbf{S} = \mathbf{CVC}'$ and the result follows by noting that the matrix product $\mathbf{C}'\left[\mathbf{CVC}'\right]^-\mathbf{C}$ is invariant for any choice of the generalized inverse where $rank\left(\mathbf{C}'\left[\mathbf{CVC}'\right]^-\mathbf{C}\right) = rank(\mathbf{C})$ if $rank\left(\mathbf{CVC}'\right) = rank(\mathbf{C})$ which follows from the fact that $\mathbf{V}$ is of full rank (see e.g. Rao & Mitra (1971), Lemma 2.2.6). Therefore, the result is true for any choice of the generalized inverse $[\mathbf{CVC}']^-$. (*Remark:* If $\mathbf{C}$ is symmetric and if $\mathbf{CVC}' = \mathbf{C}$, then $N\widetilde{\mathbf{p}}'\mathbf{CC}^-\mathbf{C}\widetilde{\mathbf{p}} = N\widetilde{\mathbf{p}}'\mathbf{C}\widetilde{\mathbf{p}}$ has asymptotically a central χ_f^2-distribution with $f = rank(\mathbf{C})$ d.f. under $H_0^F : \mathbf{CF} = \mathbf{0}$.)

(3) From Theorem 2.3 it follows that $\|\mathbf{V}-\widehat{\mathbf{V}}\| \xrightarrow{p} 0$ and that $\|\mathbf{V}^{-1}-\widehat{\mathbf{V}}^{-1}\| \xrightarrow{p} 0$. Thus, $\left\|(\mathbf{CVC}')^{-1}-(\mathbf{C}\widehat{\mathbf{V}}\mathbf{C}')^{-1}\right\| \xrightarrow{p} 0$ since $\mathbf{C}$ is of full row rank which proves the statement.

To prove (4), we note that $\mathbf{W}$ is symmetric and it follows from (2) and Lemma A.1 that $Q(\mathbf{W}) = N\widetilde{\mathbf{p}}'\mathbf{W}(\mathbf{WVW})^-\mathbf{W}\widetilde{\mathbf{p}} = N\widetilde{\mathbf{p}}'\mathbf{W}\widetilde{\mathbf{p}}$ has asymptotically a central χ_f^2-distribution with $f = rank(\mathbf{W})$ d.f. under $H_0^F : \mathbf{WF} = \mathbf{0}$. Since $\mathbf{V}$ is regular by assumption, it follows from (3) and Theorem 2.3 that $\|\widehat{\mathbf{W}}-\mathbf{W}\| \xrightarrow{p} 0$ and the result follows. □

Proof of Theorem 2.5

First it will be shown that $\sqrt{N}\mathbf{C}\widetilde{\mathbf{p}} = \sqrt{N}\mathbf{C}\int\widehat{H}d\widetilde{\mathbf{F}} \doteq \sqrt{N}\mathbf{C}\int Hd\widetilde{\mathbf{F}} = \sqrt{N}\mathbf{C}\widetilde{\mathbf{Y}}_{\cdot\cdot}$ for the sequence $\mathbf{F}_N$ defined in (9). Consider the difference

$$\sqrt{N}\mathbf{C}\int(\widehat{H}-H_N)d\widetilde{\mathbf{F}} = \sqrt{N}\mathbf{C}\int(\widehat{H}-H_N)d(\widetilde{\mathbf{F}}-\mathbf{F}_N)+\sqrt{N}\mathbf{C}\int(\widehat{H}-H_N)d\mathbf{F}_N\ .$$

Arguing as in the the proof of Theorem 2.2, it follows that

$$\sqrt{N}\mathbf{C}\int(\widehat{H}-H_N)d(\widetilde{\mathbf{F}}-\mathbf{F}_N) \quad \xrightarrow{p} \quad \mathbf{0}\ .$$

Moreover, $\sqrt{N}\mathbf{C}\int(\widehat{H}-H_N)d\mathbf{F}_N = \mathbf{C}\int(\widehat{H}-H_N)d\mathbf{K} \xrightarrow{p} \mathbf{0}$ since

$$E\left(\int(\widehat{H}-H_N)dK_{ij}\right)^2 \leq \int E(\widehat{H}-H_N)^2 dK_{ij} \leq \frac{1}{N}$$

by Jensen's inequality and by Lemma B.2,(1). Thus,

$$\sqrt{N}\mathbf{C}\int \widehat{H}d\widetilde{\mathbf{F}} \quad \doteq \quad \sqrt{N}\mathbf{C}\int H_N d\widetilde{\mathbf{F}} = \sqrt{N}\mathbf{C}\int H d\widetilde{\mathbf{F}} - \mathbf{C}\int(H-G)d\widetilde{\mathbf{F}}$$

where $G = N^{-1}\sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{k=1}^{n_i} m_{ijk}K_{ij}$. Consider now the (i,j)-component $\widetilde{F}_{ij}$ of $\widetilde{\mathbf{F}}$. Let $Z_{ijk} = m_{ijk}^{-1}\sum_{s=1}^{m_{ijk}}(H(X_{ijks})-G(X_{ijks}))$. Then it follows that

$$Var\left(\int(H-G)d\widetilde{F}_{ij}\right) = \frac{1}{n_i^2}\sum_{k=1}^{n_i} Var(Z_{ijk}) \leq \frac{1}{n_i} \to 0 \quad (\text{as } \min n_i \to \infty)$$

since Z_{ijk} and $Z_{ijk'}$ are independent for $k \neq k'$ and $|Z_{ijk}| \leq 1$. Moreover,

$$\begin{aligned} E\left(\int(H-G)d\widetilde{F}_{ij}\right) &= \frac{1}{n_i}\sum_{k=1}^{n_i}\left[\left(1-\frac{1}{\sqrt{N}}\right)\int(H-G)dF_{ij} + \frac{1}{\sqrt{N}}\int(H-G)dK_{ij}\right] \\ &= \int(H-G)dF_{ij} + O\left(\frac{1}{\sqrt{N}}\right)\ , \quad \text{as } \min n_i \to \infty. \end{aligned}$$

Thus, $E(\mathbf{C}\int(H-G)d\widetilde{\mathbf{F}}) \to \mathbf{C}\int(H-G)d\mathbf{F} = \mathbf{0}$ as $\min n_i \to \infty$. Finally, it follows that $\sqrt{N}\mathbf{C}\int \widehat{H}d\widetilde{\mathbf{F}} \doteq \sqrt{N}\mathbf{C}\widetilde{\mathbf{Y}}_{\cdot\cdot}$ which has asymptotically a multivariate normal distribution with mean $\boldsymbol{\nu}$ and covariance matrix $\mathbf{CVC}'$ since $\boldsymbol{\nu} = \sqrt{N}\mathbf{C}\int Hd\mathbf{F}_N = \int Hd(\mathbf{CK})$ and the result stated in (1) follows. Statement (2) follows immediately from (1). □

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Table 1

Subject	Probe Word 1	2	3	4	5
1	55	40	54	36.5	50
2	21	6	17	3	21
3	43.5	9.5	48.5	43.5	27
4	50.5	40	32	35	21
5	21	4	34	1	25
6	52.5	32	52.5	36.5	47
7	48.5	9.5	40	13.5	45
8	45.5	8	29	6	2
9	40	11	21	13.5	24
10	17	29	29	32	40
11	25.5	6	13.5	17	13.5
$\overline{R}_{\cdot j}$	382	177	337	216	288

Table 1. Ranks R_{kj} and rank means $\overline{R}_{\cdot j}$ for the probe word data.

Table 2

Group	Time points 1	2	3	4	5	6	7	8
control	185.3	99.6	47.7	42.6	79.0	109.6	122.7	170.7
obese	212.0	180.0	150.5	123.5	95.1	115.3	144.0	172.3

Table 2. Rank means for the PIP data.

Table 3

	Diet (E)		Diet (C)		Results		
Pair	gas O	gas N	gas O	gas N	Effect	Statistic	p-value
1	30	7	26.5	12			
2	26.5	5	19	1	Diet effect	$Q_n^A = 2.19$	0.182
3	21	22	28	3	Gas effect	$Q_n^B = 46.62$	0.00025
4	29	31	32	15	Interaction	$Q_n^{AB} = 6.47$	0.0385
5	23	13.5	18	4			
6	11	13.5	25	2			
7	9	10	24	8			
8	17	16	20	6			
mean	20.81	14.75	24.06	6.38			

Table 3. Ranks, rank means and results for the diet data.

Table 4

level	Simulated type I error $n = 7, 10$
20%	0.189−0.208
10%	0.092−0.110
5%	0.046−0.056
1%	0.007−0.013

Table 4. Range of the simulated type I error probability for the distributions (1) - (4) for $n = 7, 10$ and for $c = 0, 1, 2$. The RT of the statistic for the paired t-test is compared with the $F_{1,n-1}$-distribution.

Table 5

Simulated type I error				
	F-distribution	χ^2_{a-1}-distribution		
level	$n = 5, 10$	$n = 5$	$n = 10$	$n = 20$
20%	0.189−0.213	0.236−0.245	0.216−0.230	0.204−0.215
10%	0.094−0.108	0.139−0.152	0.116−0.124	0.106−0.113
5%	0.044−0.058	0.072−0.090	0.061−0.072	0.055−0.062
1%	0.009−0.016	0.029−0.035	0.017−0.022	0.014−0.016

Table 5. Range of the simulated type I error probability for the distributions (1) - (4). The statistic Q_N^H given in (15) is compared with the F-distribution with $f_1 = a - 1$ and $f_2 = n. - 1$ d.f. and with the asymptotic χ^2-distribution.

Table 6

Simulated type I error / F-Approximation		
level	$n = 6, 10 \quad b = 4$	$n = 15, 20 \quad b = 10$
20%	0.188−0.213	0.184−0.206
10%	0.095−0.108	0.084−0.111
5%	0.042−0.057	0.032−0.055
1%	0.006−0.015	0.004−0.013

Table 6. Range of the simulated type I error probability for the distributions (1) to (4). The statistic $(n-b+1)Q_n^M(B)/[(b-1)(n-1)]$ is compared with the $F_{b-1,n-b+1}$-distribution where $Q_n^M(B)$ is given in (13).

Table 7

Simulated type I error				
χ^2_{b-1}-Approximation, $b = 4$				
level	$n = 6$	$n = 10$	$n = 20$	$n = 30$
20%	0.513	0.400	0.284	0.245
10%	0.423	0.284	0.175	0.137
5%	0.350	0.207	0.106	0.078
1%	0.249	0.109	0.040	0.024

Table 7. Range of the simulated type I error probability for normally distributed errors and subject effects. The statistic $Q_n^M(B)$ given in (13) is compared with the asymptotic χ^2_{b-1}-distribution.

Table 8

level	Simulated type I error $n = 7, 10$
20%	0.177−0.215
10%	0.080−0.115
5%	0.027−0.057
1%	0.004−0.014

Table 8. Range of the simulated type I error probability for the distributions (1) - (4) for $n = 7, 10$ and for $c = 0, 1, 2$. The statistics $Q_n(A)$ and $Q_n(AB)$ are compared with the $F_{1,n-1}$-distribution.

Table 9

Simulated type I error / F-Approximation		
level	$n = 7, 10 \quad b = 4$	$n = 15, 20 \quad b = 10$
20%	0.181−0.212	0.188−0.234
10%	0.078−0.106	0.085−0.112
5%	0.033−0.052	0.036−0.057
1%	0.004−0.013	0.004−0.009

Table 9. Range of the simulated type I error probability for the distributions (1) to (4). The statistics F_B and F_{AB} are compared with the $F_{b-1,2n-b}$-distribution.

Table 10

Simulated type I error				
χ^2_{b-1}-Approximation, $b = 4$				
level	$n = 7$	$n = 10$	$n = 20$	$n = 30$
20%	0.322	0.290	0.243	0.228
10%	0.208	0.170	0.126	0.122
5%	0.146	0.114	0.070	0.069
1%	0.067	0.047	0.020	0.015

Table 10. Range of the simulated type I error probability for normally distributed errors and subject effects. The statistics $Q_n(B)$ and $Q_n(AB)$ given in (19) and (21), respectively, are compared with the limiting χ^2_{b-1}-distribution.