The Determinant of a Hypergeometric Period Matrix and a Generalization of Selberg's Integral

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Abstract

In an earlier paper (Adv. Appl. Math. 29 (2002), 137–151) on the determinants of certain period matrices, we formulated a conjecture about the determinant of a certain hypergeometric matrix. In this article, we establish this conjecture by constructing a system of linear equations in which that determinant is one of the variables. As a consequence, we obtain the value of an integral which generalizes the well-known multidimensional beta integral of A. Selberg (*Norsk. Mat. Tidsskr.* **26**, 71–78) and some hypergeometric determinant formulas of A. Varchenko (*Izv. Akad. Nauk SSSR Ser. Mat.* **53** (1989), 1206–1235; **54** (1990), 146–158).

Key words: Hypergeometric period matrix, multidimensional hypergeometric integral, Selberg's integral, Vandermonde determinant.

1 Introduction

In our previous article [2], we studied certain hypergeometric period matrices of Varchenko [5], [6], obtaining generalizations of the determinants of those

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matrices. In [2], we also gave applications to the well-known multidimensional beta integrals of Selberg [4] and Aomoto [1], generalized those results and provided elementary proofs thereof. Subsequently, we further generalized in [3] the results of [1], [4], [5], and [6], obtaining closed-form determinant formulas for period matrices having entries as even more general multidimensional integrals of hypergeometric type.

In the course of establishing in [2] some power-function variations on the results of Varchenko [5], we formulated a conjecture about a determinant of a hypergeometric matrix whose entries are multidimensional hypergeometric-type integrals with integrands containing power functions, and we proved in [2] some special cases of this conjecture. In the present paper, we establish the conjecture in complete generality. In order to state our result, we introduce some notation as follows.

Throughout the paper, our notation generally is consistent with [2]. Thus, for a positive integer $N \geq 2$, let $\lambda_1, \ldots, \lambda_{N+1} \in \mathbb{R}$ and $\alpha_1, \ldots, \alpha_{N+1} \in \mathbb{C}$ such that $\mathrm{Re}(\alpha_i) > 0$, $i = 1, \ldots, N+1$; and define

$$\Delta_{N+1}(\lambda_1, \ldots, \lambda_{N+1}; \alpha_1, \ldots, \alpha_{N+1}) := \prod_{1 \leq i < j \leq N+1} (\lambda_j - \lambda_i) \prod_{1 \leq i \neq j \leq N+1} (\lambda_j - \lambda_i)^{\alpha_i - 1}.$$

We shall also need the standard notation $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) = \Gamma(\alpha+k)/\Gamma(\alpha)$ for the rising factorial, where $\alpha \in \mathbb{C}$ and $k = 0, 1, 2, \ldots$.

We now state our main result.

Theorem 1.1 *For $k = 0, 1, 2, \ldots$, denote by $E_{N,k}(\lambda_1, \ldots, \lambda_{N+1}; \alpha_1, \ldots, \alpha_{N+1})$ the determinant of the $N \times N$ matrix whose (i,j)th entry is*

$$a_{i,j} = \int_{\lambda_i}^{\lambda_{i+1}} \prod_{p=1}^{N+1} (x - \lambda_p)^{\alpha_p - 1} \, x^{j-1+k} \, dx. \tag{1.1}$$

Then

$$\begin{aligned} E_{N,k}(\lambda_1, \ldots, \lambda_{N+1}; \alpha_1, \ldots, \alpha_{N+1}) = & \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_{N+1})}{\Gamma(\alpha_1 + \cdots + \alpha_{N+1})} \\ & \times \Delta_{N+1}(\lambda_1, \ldots, \lambda_{N+1}; \alpha_1, \ldots, \alpha_{N+1}) \\ & \times R_{N,k}(\lambda_1, \ldots, \lambda_{N+1}; \alpha_1, \ldots, \alpha_{N+1}) \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} & R_{N,k}(\lambda_1, \ldots, \lambda_{N+1}; \alpha_1, \ldots, \alpha_{N+1}) \\ & \qquad = \frac{1}{(\sum_{p=1}^{N+1} \alpha_p)_k} \sum_{i_1 + \cdots + i_{N+1} = k} \frac{k!}{i_1! \cdots i_{N+1}!} \prod_{p=1}^{N+1} (\alpha_p)_{i_p} \lambda_p^{k - i_p}. \end{aligned} \tag{1.3}$$

In (1.2), the principal branch of each term of the form x^{α_p-1} is fixed by $-\pi/2 < x < 3\pi/2$ for all $p = 1, \dots, N+1$. Also, in (1.3), the sum is over all nonnegative integral $i_1, \dots, i_{N+1}$ satisfying the stated condition.

For $k = 0$, Theorem 1.1 reduces to a formula of Varchenko [5], *viz.*,

$$\det\left(\int_{\lambda_i}^{\lambda_{i+1}} \prod_{p=1}^{N+1} (x-\lambda_p)^{\alpha_p-1} \; x^{j-1}\,dx\right)_{1\le i,j\le N} = \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_{N+1})}{\Gamma(\alpha_1+\cdots+\alpha_{N+1})}\,\Delta_{N+1}(\lambda_1,\dots,\lambda_{N+1};\alpha_1,\dots,\alpha_{N+1}). \quad (1.4)$$

In our paper [2], we established Theorem 1.1 for the case in which $k = 1, \dots, 5$. Here, we obtain a proof of the general case. As a consequence of the theorem, we immediately obtain extensions of Selberg's and Varchenko's formulas.

2 The Proof of Theorem 1.1

For integers $N \ge 1$, $0 \le k_1 \le k_2 \cdots \le k_N \le k$, define

$$E_{(k_1,\dots,k_N)} \equiv E_{(k_1,\dots,k_N)}(\lambda_1,\dots,\lambda_{N+1};\alpha_1,\dots,\alpha_{N+1}) := \det\left(\int_{\lambda_i}^{\lambda_{i+1}} \prod_{1\le p\le N+1} (x-\lambda_p)^{\alpha_p-1} x^{j+k_j-1}\,dx\right)_{1\le i,j\le N}.$$

Lemma 2.1 *Let k, N be natural numbers, and let $k_1, \dots, k_N$ and $i_1, \dots, i_{N+1}$ be nonnegative integers. Then the sets*

$$\boldsymbol{K} = \{(k_1,\dots,k_N) : 0 \le k_1 \le k_2 \cdots \le k_N \le k\}$$

and

$$\boldsymbol{I} = \Big\{(i_1,\dots,i_{N+1}) : \sum_{p=1}^{N+1} i_p = k\Big\}.$$

each have cardinality $\binom{k+N}{N} = (k+N)!/k!\,N!$.

The above result is classical, and we have stated it formally for direct reference at a later stage. We establish a bijection between the sets $\boldsymbol{I}$ and $\boldsymbol{K}$ by associating with each $(k_1, \dots, k_N) \in \boldsymbol{K}$ a unique $(i_1, \dots, i_{N+1}) \in \boldsymbol{I}$ by way of the substitutions

$$i_1 = k_1,\ i_2 = k_2 - k_1,\ \dots,\ i_N = k_N - k_{N-1},\ i_{N+1} = k - k_N. \quad (2.1)$$

Once this bijection is established, the cardinality of $\boldsymbol{I}$ is obtained by classical combinatorial methods.

In the sequel, we denote by $V_N(\lambda_1,\ldots,\lambda_{N+1};\alpha_1,\ldots,\alpha_{N+1})$ the right-hand side of (1.4). We now have the following result.

Lemma 2.2 *For* $(i_1,\ldots i_{N+1})\in \boldsymbol{I}$,

$$\det\left(\int_{\lambda_i}^{\lambda_{i+1}}\prod_{p=1}^{N+1}(x-\lambda_p)^{\alpha_p-1}x^{j-1}\prod_{p=1}^{N+1}(x-\lambda_p)^{i_p}\,dx\right)_{1\le i,j\le N}$$
$$=\frac{(\alpha_1)_{i_1}\cdots(\alpha_{N+1})_{i_{N+1}}}{(\alpha_1+\cdots+\alpha_{N+1})_k}\prod_{1\le p\ne q\le N+1}(\lambda_q-\lambda_p)^{i_p}$$
$$\times V_N(\lambda_1,\ldots,\lambda_{N+1};\alpha_1,\ldots,\alpha_{N+1}). \quad (2.2)$$

The proof of this result follows immediately from (1.4), on noting that the determinant in (2.2) is of the form $V_N(\lambda_1,\ldots,\lambda_{N+1};\alpha_1+i_1,\ldots,\alpha_{N+1}+i_{N+1})$, hence equals

$$\frac{\Gamma(\alpha_1+i_1)\cdots\Gamma(\alpha_{N+1}+i_{N+1})}{\Gamma(\alpha_1+\cdots+\alpha_{N+1}+k)}\prod_{1\le p<q\le N+1}(\lambda_q-\lambda_p)\prod_{1\le p\ne q\le N+1}(\lambda_q-\lambda_p)^{\alpha_p+i_p-1},$$

and this reduces to (2.2) in a straightforward way.

Next, we study an expansion of the determinant on the left-hand side of (2.2). To do this, we expand $\prod_{p=1}^{N+1}(x-\lambda_p)^{i_p}$ as a polynomial in x (with coefficients which are homogeneous polynomials in $\lambda_1,\ldots,\lambda_{N+1}$). Then the (i,j)th entry of the determinant is realized as a linear combination of terms of the form

$$\int_{\lambda_i}^{\lambda_{i+1}}\prod_{1\le p\le N+1}(x-\lambda_p)^{\alpha_p-1}x^{j+k_j-1}\,dx.$$

On expanding the determinant, we find that it is equal to a linear combination of determinants of the form $E_{(k_1,\ldots,k_N)}$ where $(k_1,\ldots,k_N)\in\boldsymbol{K}$. By viewing the terms $E_{(k_1,\ldots,k_N)}$ as unknown, we then have a linear equation in $\binom{k+N}{N}$ variables. By repeating this procedure for all $(i_1,\ldots,i_{N+1})\in\boldsymbol{I}$, we derive a system of $\binom{k+N}{N}$ linear equations in $\binom{k+N}{N}$ variables.

Example 2.3 As an illustration of the foregoing, suppose that $N=2$ and $k=2$. Then

$$\boldsymbol{K}=\{(2,2),(1,2),(0,2),(1,1),(0,1),(0,0)\},$$

and

$$\boldsymbol{I}=\{(2,0,0),(0,2,0),(0,0,2),(1,1,0),(1,0,1),(0,1,1)\}.$$

Corresponding to $(i_1, i_2, i_3) = (2, 0, 0)$, say, we have

$$
\begin{aligned}
&V_2(\lambda_1, \lambda_2, \lambda_3; \alpha_1 + 2, \alpha_2, \alpha_3) \\
&\qquad\equiv \det \left(\int_{\lambda_i}^{\lambda_{i+1}} \prod_{p=1}^{3} (x - \lambda_p)^{\alpha_p - 1} x^{j-1} (x - \lambda_1)^2 \, dx \right)_{1 \le i,j \le 2} \\
&\qquad= E_{(2,2)} - 2\lambda_1 E_{(1,2)} + \lambda_1^2 E_{(0,2)} + 3\lambda_1^2 E_{(1,1)} - 2\lambda_1^3 E_{(0,1)} + \lambda_1^4 E_{(0,0)}.
\end{aligned}
$$

By repeating this process for each $(i_1, i_2, i_3) \in \boldsymbol{I}$, we obtain the following 6×6 system of linear equations:

$$
\begin{bmatrix}
1 & -2\lambda_1 & {\lambda_1}^2 & 3{\lambda_1}^2 & -2{\lambda_1}^3 & {\lambda_1}^4 \\
1 & -2\lambda_2 & {\lambda_2}^2 & 3{\lambda_2}^2 & -2{\lambda_2}^3 & {\lambda_2}^4 \\
1 & -2\lambda_3 & {\lambda_3}^2 & 3{\lambda_3}^2 & -2{\lambda_3}^3 & {\lambda_3}^4 \\
1 & -(\lambda_1 + \lambda_2) & \lambda_1\lambda_2 & {\lambda_1}^2 + \lambda_1\lambda_2 + {\lambda_2}^2 & -(\lambda_1 + \lambda_2)\lambda_1\lambda_2 & {\lambda_1}^2{\lambda_2}^2 \\
1 & -(\lambda_1 + \lambda_3) & \lambda_1\lambda_3 & {\lambda_1}^2 + \lambda_1\lambda_3 + {\lambda_3}^2 & -(\lambda_1 + \lambda_3)\lambda_1\lambda_3 & {\lambda_1}^2{\lambda_3}^2 \\
1 & -(\lambda_2 + \lambda_3) & \lambda_2\lambda_3 & {\lambda_2}^2 + \lambda_2\lambda_3 + {\lambda_3}^2 & -(\lambda_2 + \lambda_3)\lambda_2\lambda_3 & {\lambda_2}^2{\lambda_3}^2
\end{bmatrix}
\begin{bmatrix}
E_{(2,2)} \\ E_{(1,2)} \\ E_{(0,2)} \\ E_{(1,1)} \\ E_{(0,1)} \\ E_{(0,0)}
\end{bmatrix}
$$

$$
= \frac{V_2(\lambda_1, \lambda_2, \lambda_3; \alpha_1, \alpha_2, \alpha_3)}{(\alpha_1 + \alpha_2 + \alpha_3)_2}
\begin{bmatrix}
(\alpha_1)_2(\lambda_2 - \lambda_1)^2(\lambda_3 - \lambda_1)^2 \\
(\alpha_2)_2(\lambda_1 - \lambda_2)^2(\lambda_3 - \lambda_2)^2 \\
(\alpha_3)_2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2 \\
-\alpha_1\alpha_2(\lambda_2 - \lambda_1)^2(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2) \\
-\alpha_1\alpha_3(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)^2(\lambda_2 - \lambda_3) \\
-\alpha_2\alpha_3(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)^2(\lambda_1 - \lambda_3)
\end{bmatrix}. \tag{2.3}
$$

For this special case lengthy, but straightforward, calculations based on elementary row operations lead to a solution for the variable $E_{(2,2)}$, *viz.*,

$$
E_{(2,2)} = \frac{V_2(\lambda_1, \lambda_2, \lambda_3; \alpha_1, \alpha_2, \alpha_3)}{(\alpha_1 + \alpha_2 + \alpha_3)_2} r_{(2,2)}(\lambda_1, \lambda_2, \lambda_3; \alpha_1, \alpha_2, \alpha_3) \tag{2.4}
$$

where

$$
\begin{aligned}
r_{(2,2)}(\lambda_1, \lambda_2, \lambda_3; \alpha_1, \alpha_2, \alpha_3) = {} & (\alpha_3)_2\lambda_1^2\lambda_2^2 + 2\alpha_1\alpha_2\lambda_1\lambda_2\lambda_3^2 \\
& + (\alpha_2)_2\lambda_1^2\lambda_3^2 + 2\alpha_1\alpha_3\lambda_1\lambda_2^2\lambda_3 \\
& + (\alpha_1)_2\lambda_2^2\lambda_3^2 + 2\alpha_2\alpha_3\lambda_1^2\lambda_2\lambda_3,
\end{aligned}
$$

in accordance with (1.2). In the general case, it appears that a method more powerful than row operations is necessary to solve for $E_{(k,\ldots,k)}$.

We remark too that elementary row operations also lead to the complete solution to the system of equations (2.3). Similar to (2.4), we obtain

$$E_{(k_1,k_2)}(\lambda_1,\lambda_2,\lambda_3;\alpha_1,\alpha_2,\alpha_3) = \frac{V_2(\lambda_1,\lambda_2,\lambda_3;\alpha_1,\alpha_2,\alpha_3)}{(\alpha_1+\alpha_2+\alpha_3)_{k_2}} r_{(k_1,k_2)}(\lambda_1,\lambda_2,\lambda_3;\alpha_1,\alpha_2,\alpha_3)$$

for each (k_1,k_2) where $r_{(0,0)}(\lambda_1,\lambda_2,\lambda_3;\alpha_1,\alpha_2,\alpha_3)=1$, and

$$r_{(k_1,k_2)}(\lambda_1,\lambda_2,\lambda_3;\alpha_1,\alpha_2,\alpha_3) = \begin{cases} (\alpha_2+\alpha_3)\lambda_1+(\alpha_1+\alpha_3)\lambda_2+(\alpha_1+\alpha_2)\lambda_3, & (k_1,k_2)=(0,1) \\ \alpha_3\lambda_1\lambda_2+\alpha_2\lambda_1\lambda_3+\alpha_1\lambda_2\lambda_3, & (k_1,k_2)=(1,1) \\ (\alpha_1+\alpha_2)_2\lambda_3^2+[(\alpha_3)_2+2\alpha_1\alpha_2+(\alpha_1+\alpha_2)\alpha_3]\lambda_1\lambda_2 & \\ \quad +(\alpha_1+\alpha_3)_2\lambda_2^2+[(\alpha_2)_2+2\alpha_1\alpha_3+(\alpha_1+\alpha_3)\alpha_2]\lambda_1\lambda_3 & \\ \quad +(\alpha_2+\alpha_3)_2\lambda_1^2+[(\alpha_1)_2+2\alpha_2\alpha_3+(\alpha_2+\alpha_3)\alpha_1]\lambda_2\lambda_3, & (k_1,k_2)=(0,2) \\ [\alpha_1\alpha_3+(\alpha_3)_2]\lambda_1\lambda_2^2+[\alpha_1\alpha_2+(\alpha_2)_2]\lambda_1\lambda_3^2 & \\ \quad +[\alpha_2\alpha_3+(\alpha_3)_2]\lambda_1^2\lambda_2+[\alpha_1\alpha_2+(\alpha_1)_2]\lambda_2\lambda_3^2 & \\ \quad +[\alpha_2\alpha_3+(\alpha_2)_2]\lambda_1^2\lambda_3+[\alpha_1\alpha_3+(\alpha_1)_2]\lambda_2^2\lambda_3 & \\ \quad +2(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)\lambda_1\lambda_2\lambda_3, & (k_1,k_2)=(1,2) \end{cases}$$

It would be useful to have a complete solution to the corresponding system of equations in the general case, for they would lead to further generalizations of the Selberg-type integrals which we have obtained here.

Returning to the general case, we denote by $M_{N,k}$ the matrix of coefficients of the resulting system of equations. By direct algebraic manipulations, we find that the general formula for the entry in $M_{N,k}$ corresponding to the row indexed by $(i_1,\dots,i_{N+1})$ and the column indexed by $(k_1,\dots,k_N)$ is

$$\prod_{j=1}^{N}\left(\sum_{\substack{l_1+\cdots+l_{N+1}=k_j \\ l_1\le i_1,\dots,l_{N+1}\le i_{N+1}}} (-1)^{k-k_j}\prod_{p=1}^{N+1}\lambda_p^{i_p-l_p}\right) = (-1)^{Nk-k_1-k_2-\cdots-k_N}\prod_{j=1}^{N}\left(\sum_{\substack{l_1+\cdots+l_{N+1}=k_j \\ l_1\le i_1,\dots,l_{N+1}\le i_{N+1}}}\prod_{p=1}^{N+1}\lambda_p^{i_p-l_p}\right).$$

Each entry of $M_{N,k}$ is a homogeneous polynomial in $\lambda_1,\dots,\lambda_{N+1}$. Moreover, the degree of the entries in the column corresponding to $E_{(k_1,\dots,k_N)}$ all have the common value $Nk-(k_1+\cdots+k_N)$. It follows that $\det(M_{N,k})$, the determinant of $M_{N,k}$, also is a homogeneous polynomial in $\lambda_1,\dots,\lambda_{N+1}$, and of degree

$$\sum_{0\le k_1\le\cdots\le k_N\le k}(Nk-k_1-\cdots-k_N). \tag{2.5}$$

Proposition 2.4 *The determinant of $M_{N,k}$ is a homogeneous polynomial in $\lambda_1, \dots \lambda_{N+1}$, of degree $\binom{N+1}{2}\binom{k+N}{N+1}$, and is equal to a constant multiple of*

$$\prod_{1 \le p \ne q \le N+1} (\lambda_q - \lambda_p)^{\binom{k+N}{N+1}}. \tag{2.6}$$

After the row corresponding to $(i_1, \dots, i_{N+1}) \in \boldsymbol{K}$ and the column corresponding to $E_{(k,\cdots k)}$ are eliminated from $M_{(N,k)}$, the remaining submatrix has determinant equal to a constant depending only on $(i_1, \dots, i_{N+1})$ times

$$\prod_{1 \le p \ne q \le N+1} (\lambda_q - \lambda_p)^{\binom{k+N}{N+1} - i_p} \prod_{1 \le p \le N+1} \lambda_p^{k - i_p}. \tag{2.7}$$

This proposition is the key to the proof of Theorem 1.1. Once the proposition is proved, we shall deduce from Cramer's rule that

$$E_{(k,\dots,k)} = \frac{V_N(\lambda_1, \dots, \lambda_{N+1}; \alpha_1, \dots, \alpha_{N+1})}{(\alpha_1 + \cdots + \alpha_{N+1})_k} \times \sum_{i_1 + \cdots + i_{N+1} = k} c_{(i_1, \dots, i_{N+1})} \prod_{p=1}^{N+1} (\alpha_p)_{i_p} \lambda_p^{k - i_p} \tag{2.8}$$

where the constant $c_{(i_1,\dots,i_{N+1})}$ is dependent only on $(i_1, \dots, i_{N+1})$, and then we shall complete the proof of Theorem 1.1 by calculating the constant $c_{(i_1,\dots,i_{N+1})}$.

It suffices to set $\alpha_j = 1$ for all j to deduce the value of $c_{(i_1,\dots,i_{N+1})}$ because this constant is not dependent on $\alpha_1, \dots, \alpha_{N+1}$.

Lemma 2.5 *For $\lambda_1, \dots, \lambda_{N+1} \in \mathbb{R}$,*

$$E_{(k,\dots,k)}(\lambda_1, \dots, \lambda_{N+1}; 1, \dots, 1) = \frac{1}{(k+1)_N} \prod_{1 \le i < j \le N+1} (\lambda_j - \lambda_i) \cdot \left(\sum_{i_1 + \cdots + i_{N+1} = k} \prod_{p=1}^{N+1} \lambda_p^{k - i_p} \right). \tag{2.9}$$

Moreover,

$$c_{(i_1,\dots,i_{N+1})} = \frac{k!}{i_1! i_2! \cdots i_{N+1}!}. \tag{2.10}$$

Proof. By (1.1),

$$E_{(k,\dots,k)}(\lambda_1, \dots, \lambda_{N+1}; 1, \dots, 1) = \det \left(\int_{\lambda_i}^{\lambda_{i+1}} x^{j-1+k} \, dx \right)_{1 \le i,j \le N} = \frac{1}{(k+1)_N} \det \left(\lambda_{i+1}^{j+k} - \lambda_i^{j+k} \right)_{1 \le i,j \le N}.$$

Applying elementary row operations to the latter determinant, we obtain

$$E_{(k,\dots,k)}(\lambda_1,\dots,\lambda_{N+1};1,\dots,1)=\frac{1}{(k+1)_N}\begin{vmatrix} 1 & \lambda_1^{k+1} & \cdots & \lambda_1^{k+N} \\ 1 & \lambda_2^{k+1} & \cdots & \lambda_2^{k+N} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \lambda_{N+1}^{k+1} & \cdots & \lambda_{N+1}^{k+N} \end{vmatrix}.$$

Using induction on N to evaluate the above determinant, we deduce (2.9).

Next, we set $\alpha_j = 1$ for all j in (2.8) and compare the resulting expression with (2.9). Because $\lambda_1,\dots,\lambda_{N+1}$ are arbitrary, we may compare corresponding coefficients of monomials in $\lambda_1,\dots,\lambda_{N+1}$, and then we deduce (2.10). □

It is now immediate that Theorem 1.1 follows from Proposition 2.4 and Lemma 2.5, and it remains only to establish that proposition.

3 The proof of Proposition 2.4

We show, first, that the degree of homogeneity of $\det(M_{(N,k)})$ is $\binom{N+1}{2}\binom{k+N}{N+1}$. In the sum (2.5), we replace each k_j by $k-k_j$, obtaining

$$\sum_{0\le k_N\le\cdots\le k_1\le k}(k_1+\cdots+k_N)$$

and it is noted that the order of the k_j have been reversed; we denote this latter sum by $S_N(k)$, so that $S_N(k)$ is the degree of homogeneity of $\det(M_{(N,k)})$. Applying symmetry and by comparing with (2.5), we deduce that

$$S_N(k)=\sum_{0\le k_1\le\cdots\le k_N\le k}(k_1+\cdots+k_N)=\Big(\sum_{0\le k_1\le\cdots\le k_N\le k}Nk\Big)-S_N(k).$$

Therefore

$$S_N(k)=\frac{1}{2}Nk\sum_{0\le k_1\le\cdots\le k_N\le k}1=\frac{1}{2}Nk\binom{k+N}{k},$$

which is easily rewritten in the form stated in (2.5).

Next, we prove (2.6).

For $\lambda_1=\lambda_2$, the rows corresponding to the $(N+1)$-tuples $(1,0,i_3,\dots,i_{N+1})$ and $(0,1,i_3,\dots,i_{N+1})$, where $i_3+\cdots+i_{N+1}=k-1$, are the same. Hence, we can apply elementary row operations to these two rows to extract a factor of $\lambda_2-\lambda_1$ from the determinant. By Lemma 2.1, the cardinality of the set $\{(i_3, \ldots, i_{N+1}) : i_3 + \cdots + i_{N+1} = k-1\}$ is $\binom{k+N-3}{N-2}$, and therefore we can extract in total $(\lambda_2 - \lambda_1)^{\binom{k+N-3}{N-2}}$ from such rows.

If $\lambda_1 = \lambda_2$ then the rows corresponding to the $(N+1)$-tuples $(2, 0, i_3, \ldots, i_{N+1})$, $(0, 2, i_3, \ldots, i_{N+1})$, and $(1, 1, i_3, \ldots, i_{N+1})$ are equal, where $i_3 + \cdots + i_{N+1} = k-2$. By elementary row operations, we can extract $(\lambda_2 - \lambda_1)^{\binom{3}{2}}$ from three such rows. Again by Lemma 2.1, the set $\{(i_3, \ldots, i_{N+1}) : i_3 + \cdots + i_{N+1} = k-2\}$ has cardinality $\binom{k+N-4}{N-2}$, so we can extract in total the term $(\lambda_2 - \lambda_1)^{\binom{3}{2}\binom{k+N-4}{N-2}}$ from such rows.

In the general case, if $\lambda_1 = \lambda_2$ then, for any $0 \leq n \leq k$, the following $n+1$ rows are identical: $(n, 0, i_3, \ldots, i_{N+1})$, $(n-1, 1, i_3, \ldots, i_{N+1})$, …, $(0, n, i_3, \ldots, i_{N+1})$, where $i_3 + \cdots + i_{N+1} = k - n$. Hence, using elementary row operations, we extract $(\lambda_2 - \lambda_1)^{\binom{n+1}{2}}$ from each of these $n+1$ rows. Again by Lemma 2.1, the cardinality of the set $\{(i_3, \ldots, i_{N+1}) : i_3 + \cdots + i_{N+1} = k - n\}$ is $\binom{k-n+N-2}{N-2}$, the total power of $\lambda_2 - \lambda_1$ extracted from such rows is $\binom{n+1}{2}\binom{k-n+N-2}{N-2}$.

Therefore, the power of $\lambda_2 - \lambda_1$ appearing in $\det(M_{N,k})$ is at least

$$\sum_{n=0}^{k} \binom{n+1}{2}\binom{k-n+N-2}{N-2} \equiv \binom{k+N}{N+1},$$

an identity which can be proved by induction on k.

By repeating the foregoing argument, we deduce that, for any $p \neq q$, $\det(M_{N,k})$ is divisible by $(\lambda_q - \lambda_p)^{\binom{k+N}{N+1}}$. Since these factors are relatively prime for distinct (p, q) pairs, it follows that $\det(M_{N,k})$ is divisible by (2.6). Since the degree of the polynomial (2.6) equals the degree of $\det(M_{N,k})$, then it follows that $\det(M_{N,k})$ is a constant multiple of (2.6).

Turning to the submatrix constructed by eliminating from $M_{N,k}$ the column corresponding to $E_{(k,\ldots,k)}$ and the row corresponding to $(i_1, \ldots, i_{N+1})$, we denote the determinant of the remaining submatrix by $A_{(k,\cdots,k)}^{(i_1,\ldots,i_{N+1})}$. Proceeding with a similar extraction of factors $\lambda_q - \lambda_p$, we deduce that $A_{(k,\cdots,k)}^{(i_1,\ldots,i_{N+1})}$ is divisible by

$$\prod_{1 \leq p \neq q \leq N+1} (\lambda_q - \lambda_p)^{\binom{k+N}{N+1}} \prod_{1 \leq p \neq q \leq N+1} (\lambda_q - \lambda_p)^{-i_p}$$

because, after the row $(i_1, \ldots, i_{N+1})$ is eliminated, the number of factors $(\lambda_q - \lambda_p)$ which can be extracted from the determinant is reduced by $i_p + i_q$.

We now use a highest-power argument to show that $A_{(k,\cdots,k)}^{(i_1,\ldots,i_{N+1})}$ is a constant multiple of

$$\prod_{1\le p\ne q\le N+1}(\lambda_q-\lambda_p)^{\binom{k+N}{N+1}}\prod_{1\le p\ne q\le N+1}(\lambda_q-\lambda_p)^{-i_p}\prod_{p=1}^{N+1}{\lambda_p}^{k-i_p}. \tag{3.1}$$

Because the column eliminated consists entirely of 1's, then by multiplying along the main diagonal of the subdeterminant, we find that the degree of $A_{(k,\cdots,k)}^{(i_1,\ldots,i_{N+1})}$ remains at $\binom{N+1}{2}\binom{k+N}{N+1}$. Since the degree of

$$\prod_{1\le p\ne q\le N+1}(\lambda_q-\lambda_p)^{\binom{k+N}{N+1}}\prod_{1\le p\ne q\le N+1}(\lambda_q-\lambda_p)^{-i_p} \tag{3.2}$$

is $\binom{N+1}{2}\binom{k+N}{N+1}-Nk$. Therefore the quotient of $A_{(k,\cdots,k)}^{(i_1,\ldots,i_{N+1})}$ divided by (3.2) is a polynomial homogeneous in $\lambda_1,\ldots,\lambda_{N+1}$ with degree Nk.

For each $1\le p\le N+1$, the highest power of λ_p in $\det(M_{N,k})$ is $N\binom{k+N}{N+1}$. After the column corresponding to $E_{(k,\cdots,k)}$ and the row corresponding to $(i_1,\ldots,i_{N+1})$ are eliminated from $M_{N,k}$, the highest power of λ_p in $A_{(k,\cdots,k)}^{(i_1,\ldots,i_{N+1})}$ is reduced by Ni_p. To see this, we first determine that the highest power of λ_p in row $(i_1,\ldots,i_{N+1})$ is Ni_p. In this row, there are $\binom{k+N-i_p}{N}$ entries with such power. In total, there are $\binom{k+N-1-i_p}{N-1}$ rows in which the highest power of λ_p is Ni_p. Therefore there are exactly $\binom{k+N-i_p}{N}$ columns and rows in which the highest power of λ_p is at least Ni_p for $1\le p\le N+1$.

To obtain the highest power of λ_p in the determinant, we choose one entry with power Nk, N entries with $N(k-1)$, $\binom{N+1}{2}$ entries with power $N(k-2)$, $\ldots$, $\binom{k+N-1-i_p}{N-1}$ entries with power Ni_p, $0\le i_p\le k$. Then the highest power of λ_p in $\det(M_{N,k})$ is

$$\sum_{i_p=0}^{k}Ni_p\binom{k+N-1-i_p}{N-1}\equiv N\binom{k+N}{N+1},$$

an identity which also can be established by induction on k.

After the column corresponding to $E_{(k,\ldots,k)}$ and the row corresponding to $(i_1,\ldots,i_{N+1})$ are eliminated from $M_{N,k}$, we find that the power of $A_{(k,\ldots,k)}^{(i_1,\ldots,i_{N+1})}$ in λ_p is reduced by Ni_p. Because the highest power of λ_p in

$$\prod_{1\le p\ne q\le N+1}(\lambda_q-\lambda_p)^{\binom{k+N}{N+1}}\prod_{1\le p\ne q\le N+1}(\lambda_q-\lambda_p)^{-i_p}$$

is $N\binom{k+N}{N+1}-Ni_p-(k-i_p)$, we find that the highest power of λ_p in the quotient of $A^{(i_1,\ldots,i_{N+1})}_{(k,\ldots,k)}$ by

$$\prod_{1\le p\ne q\le N+1}(\lambda_q-\lambda_p)^{\binom{k+N}{N+1}}\prod_{1\le p\ne q\le N+1}(\lambda_q-\lambda_p)^{-i_p}$$

is $k-i_p$. Since the quotient has degree Nk and $\sum_{p=1}^{N+1} i_p = k$, the only way in which we can obtain a homogeneous polynomial with degree Nk such that the highest power of λ_p is $k-i_p$ for all $1\le p\le N+1$ is if the quotient is a constant multiple of $\prod_{p=1}^{N+1}\lambda_p^{k-i_p}$. Therefore the quotient equals a constant multiple of $\prod_{p=1}^{N+1}\lambda_p^{k-i_p}$, and it follows that $A^{(i_1,\ldots,i_{N+1})}_{(k,\ldots,k)}$ is a constant times

$$\prod_{1\le p\ne q\le N+1}(\lambda_q-\lambda_p)^{\binom{k+N}{N+1}}\prod_{1\le p\ne q\le N+1}(\lambda_q-\lambda_p)^{-i_p}\prod_{1\le p\le N+1}\lambda_p^{k-i_p},$$

where the constant depends only on $(i_1,\ldots,i_{N+1})$. This completes the proof of Proposition 2.4.

4 A generalization of Selberg's integral formula

We can also formulate Theorem 1.1 in terms of generating functions. From (1.3), it is straightforward to show that

$$\sum_{k=0}^{\infty}(\lambda_1\cdots\lambda_{N+1})^{-k}R_{N+1,k}(\lambda_1,\ldots,\lambda_{N+1};\alpha_1,\ldots,\alpha_{N+1})\,\frac{z^k}{k!}=\prod_{p=1}^{N+1}\left(1-\frac{z}{\lambda_p}\right)^{-\alpha_p},$$

for $|z|<\min\{|\lambda_p| : 1\le p\le N+1\}$. Applying this result to (1.2), we obtain a similar generating function for $E_{N+1,k}(\lambda_1,\ldots,\lambda_{N+1};\alpha_1,\ldots,\alpha_{N+1})$, *viz.*,

$$\sum_{k=0}^{\infty}(\lambda_1\cdots\lambda_{N+1})^{-k}\Big(\sum_{p=1}^{N+1}\alpha_p\Big)_k E_{N+1,k}(\lambda_1,\ldots,\lambda_{N+1};\alpha_1,\ldots,\alpha_{N+1})\frac{z^k}{k!}$$
$$=\prod_{p=1}^{N+1}\left(1-\frac{z}{\lambda_p}\right)^{-\alpha_p}E_{N+1,0}(\lambda_1,\ldots,\lambda_{N+1};\alpha_1,\ldots,\alpha_{N+1}). \quad (4.1)$$

Another consequence of (1.3) results from consideration of the $N\times N$ matrix with (i,j)th entry

$$a_{i,j}=\int_{\lambda_i}^{\lambda_{i+1}}\prod_{p=1}^{N+1}|x-\lambda_p|^{\alpha_p-1}\,x^{j-1+k}\,dx, \quad (4.2)$$

where $\lambda_1<\cdots<\lambda_{N+1}$. Proceeding as in the proof of Proposition 3.1 of Richards and Zheng [2], we obtain the following generalization of that result.

Theorem 4.1 *Let $k \in \mathbb{N}$; $\lambda_1, \ldots, \lambda_{N+1} \in \mathbb{R}$ with $\lambda_1 < \cdots < \lambda_{N+1}$; and $\alpha_1, \ldots, \alpha_{N+1} \in \mathbb{C}$ with $\mathrm{Re}(\alpha_p) > 0$ for all $p = 1, \ldots, N+1$. Then*

$$\int \cdots \int_{\lambda_1 < x_1 < \lambda_2 < \cdots < \lambda_N < x_N < \lambda_{N+1}} \prod_{1 \le i < j \le N} (x_j - x_i) \prod_{j=1}^{N} \prod_{p=1}^{N+1} |x_j - \lambda_p|^{\alpha_p - 1} \prod_{j=1}^{N} x_j^k \, dx_j$$
$$= \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_{N+1})}{\Gamma(\alpha_1 + \cdots + \alpha_{N+1})} \prod_{1 \le i < j \le N+1} (\lambda_j - \lambda_i)^{\alpha_i + \alpha_j - 1}$$
$$\times \sum_{i_1 + \cdots + i_{N+1} = k} \frac{k!}{i_1! \cdots i_{N+1}!} \prod_{p=1}^{N+1} (\alpha_p)_{i_p} \lambda_p^{k - i_p}. \quad (4.3)$$

Applying to (4.3) the generating function (4.1), we obtain the following integral formula.

Corollary 4.2 *Suppose that $\lambda_1, \ldots, \lambda_{N+1} \in \mathbb{R}$ with $\lambda_1 < \cdots < \lambda_{N+1}$; $\alpha_1, \ldots, \alpha_{N+1} \in \mathbb{C}$ with $\mathrm{Re}(\alpha_p) > 0$ for all $p = 1, \ldots, N+1$; and $z \in \mathbb{C}$ such that $|z| < \min\{|\lambda_1|, \ldots, |\lambda_{N+1}|\}$. Then*

$$\int \cdots \int_{\lambda_1 < x_1 < \lambda_2 < \cdots < \lambda_N < x_N < \lambda_{N+1}} \left(1 - z \frac{\prod_{i=1}^{N} x_i}{\prod_{p=1}^{N+1} \lambda_p}\right)^{-\sum_{p=1}^{N+1} \alpha_p} \prod_{1 \le i < j \le N} (x_j - x_i)$$
$$\times \prod_{j=1}^{N} \prod_{p=1}^{N+1} |x_j - \lambda_p|^{\alpha_p - 1} \prod_{j=1}^{N} dx_j$$
$$= \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_{N+1})}{\Gamma(\alpha_1 + \cdots + \alpha_{N+1})} \prod_{1 \le i < j \le N+1} (\lambda_j - \lambda_i)^{\alpha_i + \alpha_j - 1} \prod_{p=1}^{N+1} \left(1 - \frac{z}{\lambda_p}\right)^{-\alpha_p}. \quad (4.4)$$

We define

$$P_{N,k}(\lambda_1, \ldots, \lambda_N; \gamma) = \sum_{i_1 + \cdots + i_N = k} \frac{k!}{i_1! \cdots i_N!} \prod_{p=1}^{N} (\gamma)_{i_p} \lambda_p^{k - i_p}$$

Theorem 4.3 *Let k be a nonnegative integer, and $\alpha, \beta, \gamma \in \mathbb{C}$ where $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, and $\mathrm{Re}(\gamma) > -\min\{1/N, (\mathrm{Re}\,\alpha)/(N-1), (\mathrm{Re}\,\beta)/(N-1)\}$. Then*

$$\int_0^1 \cdots \int_0^1 P_{N,k}(\lambda_1, \ldots, \lambda_{N+1}; \gamma) \prod_{1 \le i < j \le N} |\lambda_i - \lambda_j|^{2\gamma} \prod_{j=1}^{N} \lambda_j^{\alpha - 1} (1 - \lambda_j)^{\beta - 1} \, d\lambda_j$$
$$= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha + \beta + (2N-1)\gamma + k)}{\Gamma(\alpha + \beta + (N-1)\gamma)\Gamma(\alpha + N\gamma + k)\Gamma(\beta + N\gamma)}$$
$$\times \prod_{j=1}^{N} \frac{\Gamma(\alpha + j\gamma + k)\Gamma(\beta + j\gamma)\Gamma(j\gamma + 1)}{\Gamma(\alpha + \beta + (N + j - 1)\gamma + k)\Gamma(\gamma + 1)}. \quad (4.5)$$

Proof. Denote the integral on the left-hand side of (4.5) by $S_{N,k}(\alpha,\beta)$. Since the integrand there is symmetric in $\lambda_1,\ldots,\lambda_N$ then, by symmetry, we have

$$\begin{aligned} S_{N+1,k}(\alpha,\beta) &= (N+1)! \int\cdots\int_{0<\lambda_1<\cdots<\lambda_{N+1}<1} P_{N+1,k}(\lambda_1,\ldots,\lambda_{N+1};\gamma) \prod_{1\le i<j\le N+1} (\lambda_j-\lambda_i)^{2\gamma-1} \\ &\qquad \times \prod_{1\le i<j\le N+1} (\lambda_j-\lambda_i) \prod_{j=1}^{N+1} \lambda_j^{\alpha-1}(1-\lambda_j)^{\beta-1}\,d\lambda_j. \end{aligned} \tag{4.6}$$

Substituting $\alpha_1=\cdots=\alpha_{N+1}=\gamma$ in (4.3), we obtain

$$\begin{aligned} &P_{N+1,k}(\lambda_1,\ldots,\lambda_{N+1};\gamma) \prod_{1\le i<j\le N+1} (\lambda_j-\lambda_i)^{2\gamma-1} \\ &\qquad = \frac{\Gamma((N+1)\gamma)}{[\Gamma(\gamma)]^{N+1}} \int\cdots\int_{\lambda_1<x_1<\cdots<x_N<\lambda_{N+1}} \prod_{1\le i<j\le N} (x_j-x_i) \\ &\qquad\qquad \times \prod_{j=1}^{N}\prod_{p=1}^{N+1} |x_j-\lambda_p|^{\gamma-1} \prod_{j=1}^{N} x_j^k\,dx_j. \end{aligned} \tag{4.7}$$

Substituting (4.7) into (4.6) and interchanging the order of integration, we obtain

$$\begin{aligned} S_{N+1,k}(\alpha,\beta) &= (N+1)!\,\frac{\Gamma((N+1)\gamma)}{[\Gamma(\gamma)]^{N+1}} \\ &\qquad \times \int\cdots\int_{0<\lambda_1<x_1<\cdots<x_N<\lambda_{N+1}<1} \prod_{1\le i<j\le N} (x_j-x_i) \prod_{1\le i<j\le N+1} (\lambda_j-\lambda_i) \\ &\qquad \times \prod_{j=1}^{N}\prod_{p=1}^{N+1} |x_j-\lambda_p|^{\gamma-1} \prod_{j=1}^{N+1} \lambda_j^{\alpha-1}(1-\lambda_j)^{\beta-1}\,d\lambda_j \prod_{j=1}^{N} x_j^k\,dx_j. \end{aligned} \tag{4.8}$$

Let $x_0 \equiv 0$ and $x_{N+1} \equiv 1$; then for fixed $x_1, \ldots, x_N$, the inner integral in (4.8) is of the form (4.3). Indeed, with $\alpha_0 = \alpha$, $\alpha_1 = \cdots = \alpha_N = \gamma$, and $\alpha_{N+1} = \beta$,

$$\begin{aligned}
&\int\cdots\int_{0<\lambda_1<x_1<\cdots<x_N<\lambda_{N+1}<1} \prod_{1\le i<j\le N+1} (\lambda_j - \lambda_i) \prod_{j=1}^{N} \prod_{p=1}^{N+1} |x_j - \lambda_p|^{\gamma-1} \\
&\qquad\qquad \times \prod_{j=1}^{N+1} \lambda_j^{\alpha-1} (1-\lambda_j)^{\beta-1}\, d\lambda_j \\
&= \int\cdots\int_{x_0<\lambda_1<x_1<\cdots<x_N<\lambda_{N+1}<x_{N+1}} \prod_{1\le i<j\le N+1} (\lambda_j - \lambda_i) \prod_{p=1}^{N+1} \prod_{j=0}^{N+1} |\lambda_p - x_j|^{\alpha_j-1} \prod_{p=1}^{N+1} d\lambda_p \\
&= \frac{\Gamma(\alpha)\Gamma(\beta)[\Gamma(\gamma)]^N}{\Gamma(\alpha+\beta+N\gamma)} \prod_{0\le i<j\le N} (x_j - x_i)^{\alpha_i+\alpha_j-1} \\
&= \frac{\Gamma(\alpha)\Gamma(\beta)[\Gamma(\gamma)]^N}{\Gamma(\alpha+\beta+N\gamma)} \prod_{1\le i<j\le N} (x_j - x_i)^{2\gamma-1} \prod_{j=1}^{N} x_j^{\alpha+\gamma-1} (1-x_j)^{\beta+\gamma-1}.
\end{aligned}$$

Substituting this result into (4.8), we obtain

$$\begin{aligned}
&S_{N+1,k}(\alpha,\beta) \\
&\quad = \frac{(N+1)!\,\Gamma(\alpha)\Gamma(\beta)\Gamma((N+1)\gamma)}{\Gamma(\alpha+\beta+N\gamma)\Gamma(\gamma)} \\
&\qquad \times \int\cdots\int_{0<x_1<\cdots<x_N<1} \prod_{1\le i<j\le N} (x_j - x_i)^{2\gamma} \prod_{j=1}^{N} x_j^{\alpha+\gamma-1} (1-x_j)^{\beta+\gamma-1}\, x_j^k\, dx_j \\
&\quad = \frac{(N+1)!\,\Gamma(\alpha)\Gamma(\beta)\Gamma((N+1)\gamma)}{N!\,\Gamma(\alpha+\beta+N\gamma)\Gamma(\gamma)} \\
&\qquad \times \int_0^1 \cdots \int_0^1 \prod_{1\le i<j\le N} (x_j - x_i)^{2\gamma} \prod_{j=1}^{N} x_j^{\alpha+\gamma+k-1} (1-x_j)^{\beta+\gamma-1} dx_j.
\end{aligned}$$

This latter integral being Selberg's integral we evaluate it accordingly, and the resulting expression reduces to (4.5) when N is replaced by $N-1$. □

References

[1] K. Aomoto (1987). Jacobi polynomials associated with Selberg integrals. *SIAM J. Math. Anal.*, **18**, 545–549.

[2] D. Richards and Q. Zheng (2002). Determinants of period matrices and an application to Selberg's multidimensional beta integral. *Adv. in Appl. Math.*, **28**, 602-633.

[3] D. Richards and Q. Zheng (2002). Determinant formulas for multidimensional hypergeometric period matrices. *Adv. in Appl. Math.*, **29**, 137-151.

[4] A. Selberg (1944). Bemerkninger om et multipelt integral. *Norsk. Mat. Tidsskr.*, **26**, 71–78.

[5] A. Varchenko (1989). The Euler beta-function, the Vandermonde determinant, the Legendre equation, and critical values of linear functions on a configuration of hyperplanes. I. *Izv. Akad. Nauk SSSR Ser. Mat.*, **53**, 1206–1235 (= *Math. USSR-Izv.*, **35** (1990), 543–571).

[6] A. Varchenko (1990). The Euler beta-function, the Vandermonde determinant, the Legendre equation, and critical values of linear functions on a configuration of hyperplanes. II. *Izv. Akad. Nauk SSSR Ser. Mat.*, **54**, 146–158 (= *Math. USSR-Izv.*, **36** (1991), 155–167).