Exam 3 Solutions **STAT/MATH 418** **Fall 2007**

1. Two trucks break down at points randomly distributed on a road of length one mile. Find the expected value of the square of the distance between them. Assume independence.

   Solution: Let X and Y denote the points on the one mile stretch where the two trucks broke down. It is given that X and Y are independent and uniformly distributed on $(0,1)$. So

   $$E(X) = E(Y) = \frac{1}{2},\ E(X^2) = E(Y^2) = \int_0^1 t^2 dt = \frac{1}{3}, \quad \text{and} \quad E(XY) = E(X)E(Y) = \frac{1}{4}.$$

   $|X - Y|$ is the distance between the two trucks.

   $$E(X-Y)^2 = E(X^2 + Y^2 - 2XY) = 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6}.$$

2. Suppose that the expected number of accidents per week at an industrial plant is 5. Suppose also that the numbers of workers injured in each accident are independent random variables with a common mean of 3. If the number of workers injured in each accident is independent of the number of accidents that occur, compute the expected number of workers injured in a week.

   Solution: Let N = number of accidents in a week. $E(N) = 5$.

   Let X_i = number of workers injured in i-th accident. $E(X_i) = 3$.

   $S = \sum_{i \le N} X_i$ = number of workers injured in a week.

   $$E\left(S|N=n\right) = E\left(\sum_{i=1}^{n} X_i|N=n\right) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E\left(X_i\right) = 3n.$$

   $$E(S) = E(E(S|N)) = E(3N) = 3 \times 5 = 15.$$

3. A fair coin is tossed 100 times, and suppose Y denotes the total number of times heads turned up. Let X denote the number of tails among the first 50 tosses. Find $Var(Y|X=0)$. (*Hint*: Check if X and $Y - 50 + X$ are independent).

   Solution: As the number of head in the first 50 tosses $= 50 - X$, $\quad W = Y - (50 - X) =$ the number of heads in the last 50 tosses is independent of X. Now W has binomial distribution with $n = 50$ and $p = \frac{1}{2}$.

   $$\begin{aligned} Var(Y|X=0) &= Var(Y - 50 + 0|X=0) = Var(Y - 50 + X|X=0) \\ &= Var(W|X=0) = Var(W) = 50 \times \frac{1}{2} \times \frac{1}{2} = 12.5. \end{aligned}$$

4. The joint probability density function of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} c\,x\,e^{-x^3} & \text{if} \quad -x < y < x, \quad x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

for some $c > 0$. Find the conditional density of $Y|X = 1$. Compute $Var(Y|X = 1)$.

Solution: $f_X(x) = 2c\,x^2 e^{-x^3}$ if $x > 0$ and zero otherwise. So $f_{Y|X}(y|x) = \frac{1}{2x}$ if $x > 0, -x < y < x$ and zero otherwise. Thus $f_{Y|X}(y|1) = \frac{1}{2}$ if $-1 < y < 1$ and zero otherwise. $E(Y|X = 1) = 0$ and

$$E(Y^2|X = 1) = \int_{-1}^{1} \frac{1}{2} y^2\, dy = \frac{1}{3}.$$

So $Var(Y|X = 1) = \frac{1}{3}$.

5. Suppose X is a continuous random variable with mean and variance both equal to 10.335. Find $Corr(10.335X + 17, -6X + 10.335)$.

Solution:

$$\begin{aligned} Cov(10.335X + 17, -6X + 10.335) &= -6 \times 10.335\, Var(X), \\ Var(10.335X + 17) &= (10.335)^2\, Var(X) \\ Var(-6X + 10.335) &= 6^2\, Var(X). \end{aligned}$$

So

$$Corr(10.335X + 17, -6X + 10.335) = \frac{Cov(10.335X + 17, -6X + 10.335)}{\sqrt{Var(10.335X + 17)\, Var(-6X + 10.335)}} = -1.$$

6. Suppose X, Y are independent normal random variables with mean 0 and variance 1. Find the joint density of $(X - 2Y,\ 2X + Y)$. Are $X - 2Y$ and $2X + Y$ independent? Justify your answer.

Solution: Put $U = X - 2Y$ and $V = 2X + Y$. So $X = \frac{1}{5}(U + 2V)$ and $Y = \frac{1}{5}(V - 2U)$. Jacobian is 5.

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{2\pi 5} \exp\left\{ -\frac{1}{2}\left(\frac{u+2v}{5}\right)^2 - \frac{1}{2}\left(\frac{v-2u}{5}\right)^2 \right\} \\ &= \frac{1}{\sqrt{2\pi 5}}\, e^{-\frac{1}{10}u^2}\, \frac{1}{\sqrt{2\pi 5}}\, e^{-\frac{1}{10}v^2}. \end{aligned}$$

So U and V are independent normal variables.