Do Random phenomena exist in Nature?

Which way a coin tossed in air will fall is after all completely determined by laws of physics.

We will need to measure too many parameters to figure out the trajectory, and hence the face of the coin when it hits the ground!

If Mathematics and Probability theory were as well understood several centuries ago as they are today but the planetary motion was not understood, perhaps the occurrence of a Solar eclipse would have been modeled as a random event and could have been assigned a probability based on empirical occurrence.

Subsequently, someone would have revised the model- observing that solar eclipse occurs only on a new moon day.

Of course, as time passed, the phenomenon would be completely understood and the model changed from a random model to deterministic.

We often come across events whose outcome is uncertain.

- It may be too expensive or even counterproductive to observe all the inputs.
- The uncertainty could be due to the current level of understanding of the phenomenon.
- The uncertainty could be due to the future choices - such as outcome of an election yet to be held.
- A natural question: If the outcome is uncertain, why do we think that it is possible to model it mathematically?

Micro vs. Macro

- The events that are uncertain at micro level appear to be deterministic at macro level.
- While the sex of a child about to be conceived is uncertain, the proportion of boys to girls born in a city over a period of time - say a year is stable. This has been observed over centuries across several countries where birth records have been kept.

The same phenomenon - uncertain at micro level but almost deterministic at macro level - has been observed when it comes to weather data: rainfall, high tide levels, maximum and minimum temperatures.

Probability theory attempts to capture this phenomenon - of micro level uncertainty giving rise to deterministic behavior at macro level.

The macro level observations are a guide to construction of the model for micro level.

Randomness is in the eye of the observer

Outcomes, Sample Space, Events

- An outcome is a result of an experiment.
  - An experiment means any action that can have a number of possible results, but which result will actually occur cannot be predicted with certainty prior to the experiment. *e.g.* Tossing of a coin.
- The set of all possible outcomes of an experiment is the sample space.
- A set of outcomes or a subset of the sample space is an event.

Suppose an experiment $\mathcal{E}$ results in one of the outcomes $\{e_1, e_2, \ldots e_m\}$.

Model probability by assigning weight p_i to the outcome e_i for every i in such a way that p_i's add up to 1.

Set $\Omega = \{e_1, e_2, \ldots e_m\}$ is called sample space
Each outcome e_i is called an elementary event
A subset $A \subseteq \Omega$ is called an event

Probabilities of elementary outcomes

How to assign the probabilities to the elementary outcomes?

The simplest case is when due to inherent symmetries present, we can model all the elementary events (*i.e.* outcomes) as being equally likely.

Examples

Example 1: Toss of a coin. $\Omega = \{H, T\}$ with $P(H) = .5$ and $P(T) = .5$ This is our model for the probabilities if we believe that there is symmetry.

Example 2: Toss of a dice. $\Omega = \{1, 2, 3, 4, 5, 6\}$ with $P(i) = \frac{1}{6}$ for $i = 1, 2, \ldots 6$.

Example 3 Toss of a quarter, a dime and a nickle together. We list outcomes as $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ where first letter is outcome of the quarter, second of the dime and third of the nickle.

Here it is reasonable to model all the outcomes as equally likely.

Once we agree on this, the model then becomes

$$P(e_i) = \frac{1}{8} \text{ for all } i$$

Example 4 Toss a quarter, a dime and a nickle together, and we note the number of heads. The list of outcomes now is $\{0, 1, 2, 3\}$.

Do we have reason to model these as equally likely?

A little thought would convince us otherwise.

An experiment would also reveal the same: if we toss the three coins 100 times we would observe that $\mathbf{1, 2}$ occur far more than $\mathbf{0, 3}$.

So the events are not equally likely.

How do we assign the probabilities?

We have seen that in Example 3, all the 8 outcomes were equally likely.

Write Z for the outcome of example 3 and Y as the outcome of example 4.

Then

$Y = 0$ is same as $Z =$TTT

$Y = 1$ is same as $Z =$HTT, THT or TTH

$Y = 2$ is same as $Z =$HHT, HTH or THH

$Y = 3$ is same as $Z =$HHH.

It is reasonable to model the probabilities as $P(Y = 0) = P(Y = 3) = 0.125$, $P(Y = 1) = P(Y = 2) = 0.375$.

Even when the outcomes are not equally likely, we may be able to identify outcomes as combinations of equally likely outcomes of another experiment and thus obtain a model for the probabilities.

When an experiment $\mathcal{E}$ results in m equally likely outcomes $\{e_1, e_2, \ldots e_m\}$, probability of any event A is simply

$$P(A) = \frac{\#A}{m}$$

which is often read as ratio of number of favorable outcomes and the total number of outcomes.

How to count?

The basic principle of counting

Experiments A and B are performed.

Experiment A can result in any one of m possible outcomes.

For each outcome of experiment A there are n possible outcomes of experiment B.

Then together there are mn possible outcomes of the two experiments.

$$
\begin{array}{llll}
(1,1), & (1,2), & \ldots, & (1,n) \\
(2,1), & (2,2), & \ldots, & (2,n) \\
\vdots & & & \\
(m,1), & (m,2), & \ldots, & (m,n)
\end{array}
$$

A popular old nursery rhyme

As I was going to St. Ives,
I met a man with seven wives,
Each wife had seven sacks,
Each sack had seven cats,
Each cat had seven kits.
Kits, cats, sacks and wives,
How many were going to St. Ives?

Problem 8

How many different letter arrangements can be made from the letters

1. FLUKE – Permutations
2. PROPOSE
3. MISSISSIPPI
4. ARRANGE

Let’s Make a Deal

(Monty Hall Paradox)