COMMUN. STATIST.—THEORY METH., 25(7), 1617–1630 (1996)

SIGN TEST FOR RANKED-SET SAMPLING

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Key Words: Order statistic; Beta distribution function; Binomial distribution; Nonparametric test.

ABSTRACT

The exact null distribution of the ranked-set sample (RSS) sign test statistic is computed. The power of this test is compared with the simple random sample (SRS) sign test for some continuous symmetric distributions. The problem of imperfect judgment is discussed. The superiority of RSS over SRS is demonstrated.

1. INTRODUCTION

McIntyre (1952) introduced the ranked-set sampling (RSS) for situations where measurements are difficult but ranking of the potential sample data is relatively easy. The idea is to select k units at random from a specified population and then rank them. The unit designated as the smallest is measured. The other $k-1$ units are returned to the population and another set of k units is chosen. Again, they are ranked and the second smallest is measured. The $k-1$ remaining units are returned to the population. This is continued until k ordered units are measured. This completes the first cycle. Note that k^2 units are selected but only k of them are measured. They are

$$X_{(1)1}, X_{(2)1}, \cdots, X_{(k)1},$$

where $X_{(j)1}$ is the measurement taken on the unit judged to have rank j in the first cycle. This process is repeated for n cycles to get a ranked-set sample of nk measurements:

$$\{ X_{(j)1}, X_{(j)2}, \cdots, X_{(j)n}; \quad j = 1, 2, \cdots, k \}$$

This method of sampling can result in improved statistical inference. Non-parametric methods toward analyzing ranked-set samples are considered by Stokes and Sager (1988), Bohn and Wolfe (1992a, 1992b, 1994), Kvam and Samaniego (1994) and Hettmansperger (1995).

We suppose the population be distributed as $H(x) = F(x - \theta)$ and we wish to make statistical inferences concerning θ, such as estimation or testing. Hettmansperger (1995) derives an asymptotic RSS version of the sign test and investigates its properties. We extend this work by finding the exact null distribution of the RSS sign test statistic for small values of k and n and by comparing the efficiency of this test with the SRS sign test for double exponential, Cauchy, and contaminated normal distributions.

2. EXACT NULL DISTRIBUTION

Let $H(x) = F(x - \theta)$ denote the population distribution, where F is a distribution function with median zero. Consider the testing problem $H_0 : \theta = \theta_0$. Under the ranked-set sampling all nk measurements are independent. Therefore, $X_{(j)1}, X_{(j)2}, \cdots, X_{(j)n}$ are independent and identically distributed for each $j = 1, 2, \cdots k$. The cdf $H_{(j)}(x)$ of $X_{(j)1}$ is

$$H_{(j)}(x) = \frac{k!}{(j-1)!\,(k-j)!} \int_{-\infty}^{x} H^{j-1}(t)[1 - H(t)]^{k-j} h(t)\, dt, \qquad -\infty < t < \infty,$$

where $h(x)$ is the pdf of X. We write

$$I_x(\alpha, \beta) = \frac{1}{B(\alpha, \beta)} \int_0^x u^{\alpha-1}(1-u)^{\beta-1}\, du,$$

where $B(\alpha, \beta)$ is the beta function. We note that a change of variable $u = H(t)$ gives us $H_{(j)}(\theta) = I_{1/2}(j, k-j+1)$. See Theorem 1 in Hettmansperger (1995).

Hettmansperger (1995) observes that the appropriate RSS sign test statistic for testing $H_0 : \theta = \theta_0$ is

$$S^+_{RSS} = \sum_{j=1}^{k}\sum_{i=1}^{n} I(X_{(j)i} - \theta_0 > 0),$$

where $I(\cdot)$ is the indicator function. $E(S^+_{RSS})$ and $Var(S^+_{RSS})$ are calculated under H_0. He notes that $E(S^+_{RSS}) = nk/2$ and $Var(S^+_{RSS}) = (1/4)nk\lambda^2$, where $\lambda^2 = 1 - (4/k)\sum_j [H_{(j)}(\theta) - 1/2]^2$. Therefore, the asymptotic distribution of $(2S^+_{RSS} - nk)/\lambda\sqrt{nk}$, as $n \to \infty$, is the standard normal distribution. We present an exact RSS test for testing $H_0 : \theta = \theta_0$. The test statistic S^+_{RSS} can be written as $S^+_{RSS} = \sum_j S^+_{(j)}$, where

$$S^+_{(j)} = \sum_{i=1}^{n} I(X_{(j)i} - \theta_0 > 0)$$

has a binomial distribution with parameters n and $1 - I_{1/2}(j, k-j+1)$. Further, $S^+_{(j)}$, $j = 1, 2, \cdots, k$ are stochastically independent. Thus we have

Theorem 1: The exact null distribution of S^+_{RSS} is given by

$$P(S^+_{RSS} = y) = \sum_{\mathbf{J}_y} \prod_{r=1}^{k} \binom{n}{j_r} [1 - I_{1/2}(r, k-r+1)]^{j_r} [I_{1/2}(r, k-r+1)]^{n-j_r},$$

for y $= 0, 1, \cdots,$ nk; where

$$\mathbf{J}_y = \{(j_1, j_2, \cdots, j_k) : \sum_{r=1}^{k} j_r = y;\ 0 \le j_r \le n,\ r = 1, 2, \cdots, k\}.$$

Remark 1: The cardinality of $\mathbf{J}_y$ is

$$\begin{aligned} card(\mathbf{J}_y) &= \binom{y+k-1}{k-1} - \binom{k}{1}\binom{y+k-n-2}{k-1} + \binom{k}{2}\binom{y+k-2n-3}{k-1} \\ &- \cdots \qquad + (-1)^k \binom{k}{k}\binom{y+k-kn-(k+1)}{k-1} \end{aligned}$$

See Theorem 2 Derman and Fryer (p. 79, 1972).

The proof of the following corollary is straightforward and is omitted.

Corollary 1: An alternative expression for $P(S^+_{RSS} = y)$ is

$$P(S^+_{RSS} = y) = \left[\binom{kn}{n, \cdots, n}(1/k)^{kn}\right]^{-1} \sum_{\mathbf{J}_y} \binom{kn}{j_1, \cdots, j_k, n-j_1, \cdots, n-j_k}$$

$$\times \prod_{r=1}^{k}[(1 - I_{1/2}(r, k-r+1))/k]^{j_r}[(I_{1/2}(r, k-r+1))/k]^{n-j_r}$$

From Theorem 1 and Corollary 1 we note that, for $j_1, \cdots, j_k$ in $\mathbf{J}_y$,

$$P(S_{(1)+} = j_1, \cdots, S_{(k)+} = j_k) =$$

$$\left[\binom{kn}{n, \cdots, n}(1/k)^{kn}\right]^{-1} \binom{kn}{j_1, \cdots, j_k, n-j_1, \cdots, n-j_k}$$

$$\times \prod_{r=1}^{k}[(1 - I_{1/2}(r, k-r+1))/k]^{j_r}[I_{1/2}(r, k-r+1))/k]^{n-j_r}$$

The right hand expression is the conditional probability of a simple random sample $Y_1, \cdots, Y_{kn}$ from a population being cross-classified into a $k \times 2$ table given that the row totals $T_j = n$ for all $j = 1, \cdots, k$. This points out to the characterization of a ranked-set sample discussed in Theorem 1 in Stokes and Sager (1988). That is, the distribution theory for ranked-set sampling is the same as the distribution theory for simple random sampling conditioned on $T_1 = \cdots = T_k = n$.

The null distributions of S^+_{RSS} for some selected values of k and n are computed and are presented in the Appendix. We use the IMSL subroutines DBETDF and DBINPR to evaluate the beta distribution function $I_x(\alpha, \beta)$ and the binomial probability $P(S^+_{(r)} = j_r)$, respectively.

The equivalent simple random sample sign test statistic is

$$S^+_{SRS} = \sum_{i=1}^{nk} I(X_i - \theta_0 > 0),$$

where $X_1, \cdots, X_{kn}$ is a random sample from the population $H(x)$. The exact null distribution of the simple random sample sign test statistic S^+_{SRS} is binomial with parameters nk and $1/2$. See Gibbons (1971).

We observe that there is a significantly higher portion of the probability mass at the tail-end values of the SRS test statistic than for S^+_{RSS}. This makes the variance of S^+_{RSS} smaller than the variance of S^+_{SRS} as proved by Hettmansperger (1995).

3. THE TEST

We now state the exact test procedure. Reject the null hypothesis H_0 : $\theta = \theta_0$ in favor of the alternative hypothesis $H_1 : \theta \neq \theta_0$ if $S^+_{RSS} \leq m$ or $S^+_{RSS} \geq nk - m$, where m is read from the appropriate table in the Appendix such that $P(S^+_{RSS} \leq m) = \alpha/2$. Since S^+_{RSS} has discrete distribution, the choice of α is limited.

Confidence intervals for θ are obtained as follows. Let $X^*_{(1)}, \cdots, X^*_{(nk)}$ be the ordered values of $X_{(j)i}$, $j = 1, \cdots, k$ and $i = 1, \cdots, n$. Then the interval $[X^*_{(m+1)}, X^*_{(nk-m)}]$ is a $100(1-\alpha)\%$ confidence interval for θ where $P(S^+_{RSS} \leq m) = \alpha/2$.

4. POWER COMPARISON

For several distributions, powers of $\mathrm{S}^+_{\mathrm{RSS}}$ and $\mathrm{S}^+_{\mathrm{SRS}}$ to test $\mathrm{H}_0 : \theta = 0$ vs. $\mathrm{H}_1 : \theta = \delta (> 0)$ are compared in this section. The RSS sign test is based on k=5 and n=5. The significance level chosen for the RSS sign test is $\alpha_{RSS} = 0.0414$. For the SRS test we choose the significance level α_{SRS} to be 0.0539.

Example 1: Double Exponential Distribution

Consider the double exponential distribution:

$$h(x) = (1/2)\exp(-\mid x - \theta \mid), \quad -\infty < x < \infty.$$

We reject H_0 if $S^+_{RSS} \geq m$, where m is chosen such that $P(S^+_{RSS} \geq m) = \alpha$. Under H_1, the distribution function $H(x) = (1/2)\,exp(x - \delta)$ for $x \leq \delta$ and $H(x) = 1 - (1/2)\,exp(\delta - x)$ otherwise. It follows that

$$H_{(j)}(0) = I_{.5e^{-\delta}}(j, k - j + 1), \quad \text{for } \delta \in H_1.$$

For the RSS sign test decision rule is to reject H_0 if S^+_{RSS} is greater than or equal to 16 and for the SRS sign test decision rule is to reject H_0 if S^+_{SRS} is greater than or equal to 17. Let $\beta(\cdot)$ denote the power function of a test. Then we have

$$\beta_{SRS}(\delta) = \sum_{y=17}^{25} \binom{25}{y} [1 - .5e^{-\delta}]^y [.5e^{-\delta}]^{25-y}$$

$$\beta_{RSS}(\delta) = \sum_{y=16}^{25} \sum_{\mathbf{J}_y} \prod_{r=1}^{5} \binom{5}{j_r} [1 - I_{.5e^{-\delta}}(r, 5-r+1)]^{j_r} [I_{.5e^{-\delta}}(r, 5-r+1)]^{5-j_r}$$

They are computed and compared in Table 1(a).

Example 2: Cauchy Distribution

Consider the Cauchy distribution:

$$h(x) = \frac{1}{\pi} \frac{1}{1+(x-\theta)^2}, \quad -\infty < x < \infty.$$

Under H_1, we have

$$H(0) = \frac{1}{2} + \frac{1}{\pi} \arctan(-\delta), \text{ and } H_{(j)}(0) = I_{H(0)}(j, k-j+1).$$

We choose the same decision rules considered in Example 1. The power functions are:

$$\beta_{SRS}(\delta) = \sum_{y=17}^{25} \binom{25}{y} [\frac{1}{2} - \frac{1}{\pi} \arctan(-\delta)]^y [\frac{1}{2} + \frac{1}{\pi} \arctan(-\delta)]^{25-y}$$

$$\beta_{RSS}(\delta) = \sum_{y=16}^{25} \sum_{\mathbf{J}_y} \prod_{r=1}^{5} \binom{5}{j_r} [1 - I_{H(0)}(r, 5-r+1)]^{j_r} [I_{H(0)}(r, 5-r+1)]^{5-j_r}$$

These two power functions are computed and compared in Table 1(b).

Example 3: Contaminated Normal Distribution

Consider the following contaminated normal distribution:

$$H(x) = 0.9\Phi(x-\theta) + 0.1\Phi((x-\theta)/2),$$

where $\Phi(\cdot)$ is the standard normal distribution function. Under H_1, we have

$$H(0) = 0.9\Phi(-\delta) + 0.1\Phi(-\delta/2), \text{ and } H_{(j)}(0) = I_{H(0)}(j, k-j+1).$$

Once again we choose the decision rules considered in the earlier examples. We compute the power functions as follows.

$$\beta_{SRS}(\delta) = \sum_{y=17}^{25} \binom{25}{y} [1 - .9\Phi(-\delta) - .1\Phi(-\delta/2)]^y [.9\Phi(-\delta) + .1\Phi(-\delta/2)]^{25-y}$$

$$\beta_{RSS}(\delta) = \sum_{y=16}^{25} \sum_{J_y} \prod_{r=1}^{5} \binom{5}{j_r} [1 - I_{H(0)}(r, 5 - r + 1)]^{j_r} [I_{H(0)}(r, 5 - r + 1)]^{5-j_r}$$

The standard normal distribution function $\Phi(\cdot)$ is computed using the IMSL subroutine DNORDF. Numerical values of β_{SRS} and β_{RSS} for some values of δ are presented in Table 1(c) for comparison.

5. IMPERFECT JUDGMENT

Bohn and Wolfe (1992b, 1994) address the issue of imperfect judgment rankings in ranked-set sampling, and, in particular, their effect on the properties of test procedures based on the ranked-set samples analog of the Mann-Whitney-Wilcoxon statistic. They propose a model for the probabilities of imperfect judgment rankings. Under this model p_{ij} is the probability that the unit that actually has numerical rank i in the set is chosen as the jth judgment order statistic. For example, for $k = 5$, the following set of p_{ij}'s makes sense.

$$(p_{ij}) = \begin{bmatrix} .75 & .25 & 0 & 0 & 0 \\ .25 & .50 & .25 & 0 & 0 \\ 0 & .25 & .50 & .25 & 0 \\ 0 & 0 & .25 & .50 & .25 \\ 0 & 0 & 0 & .25 & .75 \end{bmatrix}$$

In this case, the expert has a 75% chance of getting the extremes correct, otherwise she has a 50% chance of being correct and the remaining 50% is evenly split among the immediate neighbors. The measurement of the unit judged to have rank j is denoted by $x_{[j]}$. The distribution function of $X_{[j]}$ is then given by

$$H_{[j]}(x) = \sum_{i=1}^{k} p_{ij} H_{(i)}(x),$$

where $H_{(i)}(x)$ is the distribution function of the ith order statistic $X_{(i)}$. See Hettmansperger (1995) also.

We wish to examine the performance of the ranked-set sample sign test when the ranking is not perfect. Let β_{RSS}^P denote the power of the RSS sign

Table 1: Power comparison when the set size k=5 and the number of cycles n = 5. $\beta_{SRS}(\delta)$, $\beta_{RSS}(\delta)$, and $\beta_{RSS}^{P}(\delta)$ denote the power of the test at $\theta = \theta_0+\delta$, respectively, for simple random sample, ranked-set sample and ranked-set sample under imperfect judgment model P.

(a) Double Exponential Distribution

δ	0.0	0.1	0.25	0.5	0.75	1.0	1.25
β_{SRS}	.0539	.1288	.3108	.6638	.8865	.9713	.9941
β_{RSS}	.0414	.1475	.4469	.8767	.9876	.9993	1.0
β_{RSS}^{P}	.0563	.1691	.4515	.8545	.9792	.9980	.9998

(b) Cauchy Distribution

δ	0.0	0.1	0.25	0.5	1.0	1.5	2.0
β_{SRS}	.0539	.0984	.2044	.4567	.8506	.9681	.9928
β_{RSS}	.0414	.1010	.2716	.6599	.9779	.9991	.9999
β_{RSS}^{P}	.0563	.1216	.2892	.6467	.9663	.9976	.9998

(c) Contaminated Normal Distribution

δ	0.0	0.1	0.25	0.5	0.75	1.0	1.25
β_{SRS}	.0539	.1094	.2529	.6035	.8788	.9797	.9981
β_{RSS}	.0414	.1175	.3526	.8250	.9858	.9996	1.0
β_{RSS}^{P}	.0563	.1387	.3648	.8029	.9767	.9989	1.0

test. The superscript P over β refers to the set p_{ij}'s shown above in the form of a matrix. We note that $S^+_{(j)}$ has a binomial distribution with parameters n and $1 - \sum_i p_{ij} I_{H(0)}(j, k-j+1)$. We calculate β^P_{RSS} for the distributions considered in Examples 1-3 in Section 4 and present β^P_{RSS} along with β_{SRS} and β_{RSS} in Table 1.

6. CONCLUSIONS

If the actual measurements of the sample observations are difficult but ranking of the sample data is relatively easy, then ranked-set sampling is recommended. For the same reason we may not choose a large n. Therefore, an exact RSS sign test for $H_0 : \theta = \theta_0$ is a viable solution. In addition, computing the exact null distribution of S^+_{RSS} has pedagogic value; it may help a beginner to understand why improved statistical inference can result from using ranked-set sampling techniques as opposed to the usual simple random sampling approach. Under the Bohn-Wolfe (1992b) imperfect judgment ranking model of Section 5, the RSS sign test is still more powerful than the SRS sign test except perhaps in the neighborhood of $\delta = 0$. If the ranking is not perfect, it may be possible that the RSS sign test has a higher Type I error probability than the SRS sign test. The numerical results in Table 1 demonstrate the superiority of the RSS sign test over the SRS sign test.

7. APPENDIX

Table A.1. Exact Null Distribution of S^+_{RSS}

$P(S^+_{RSS} = y)$

y	k=2,n=5	k=2,n=6	k=2,n=7	k=2,n=8	k=2,n=9	k=2,n=10
0	.00023	.00004	.00001			
1	.00386	.00087	.00019	.00004		
2	.02691	.00750	.00196	.00049	.00012	.00003
3	.10128	.03653	.01170	.00345	.00096	.00026

Table A.1. Exact Null Distribution of S^+_{RSS}

	$P(S^+_{RSS} = y)$					
y	k=2,n=5	k=2,n=6	k=2,n=7	k=2,n=8	k=2,n=9	k=2,n=10
4	.22262	.11009	.04488	.01610	.00527	.00161
5	.29020	.21254	.11550	.05190	.02044	.00730
6	.22262	.26486	.20314	.11869	.05771	.02458
7	.10128	.21254	.24524	.19460	.12040	.06248
8	.02691	.11009	.20314	.22945	.18690	.12112
9	.00386	.03653	.11550	.19460	.21638	.17996
10	.00023	.00750	.04488	.11869	.18690	.20533
11		.00087	.01170	.05190	.12040	.17996
12		.00004	.00196	.01610	.05771	.12112
13			.00019	.00345	.02044	.06248
14			.00001	.00049	.00527	.02458
15				.00004	.00096	.00730
16					.00012	.00161
17					.00001	.00026
18						.00003

Table A.2. Exact Null Distribution of S^+_{RSS}

$P(S^+_{RSS} = y)$

y	k=3,n=3	k=3,n=4	k=3,n=5	k=3,n=6	k=3,n=10
0	.000164	.000009			
1	.003995	.000291	.000020	.000001	
2	.036530	.003850	.000344	.000028	
3	.153865	.026470	.003292	.000342	
4	.305445	.101708	.019080	.002664	
5	.305445	.223238	.069498	.013782	.000003
6	.153865	.288866	.161947	.048481	.000027
7	.036530	.223238	.245817	.117552	.000180
8	.003995	.101707	.245817	.198827	.000943
9	.000164	.026470	.161947	.236644	.003898
10		.003850	.069498	.198827	.012804
11		.000291	.019080	.117552	.033623
12		.000009	.003292	.048481	.070890
13			.000344	.013782	.120438
14			.000020	.002664	.165335
15				.000342	.185335
16				.000028	.165335
17				.000001	.120438
18					.070890
19					.033623
20					.012804
21					.003898
22					.000943
23					.000180
24					.000027
25					.000003

Table A.3. Exact Null Distribution of S^+_{RSS}

	$P(S^+_{RSS} = y)$				
y	k=4,n=2	k=4,n=3	k=4,n=4	k=4,n=5	k=4,n=6
0	.000158	.000002			
1	.005617	.000106	.000002		
2	.063076	.002131	.000052	.000001	
3	.241485	.020116	.000785	.000022	.000001
4	.379326	.093247	.006811	.000289	.000009
5	.241485	.229466	.034802	.002384	.000106
6	.063076	.309864	.108901	.012911	.000852
7	.005617	.229466	.214378	.046920	.004796
8	.000158	.093247	.268539	.116625	.019296
9		.020116	.214378	.200670	.056258
10		.002131	.108901	.240355	.120076
11		.000106	.034802	.200670	.188838
12		.000002	.006811	.116625	.219534
13			.000785	.046920	.188838
14			.000052	.012911	.120076
15			.000002	.002384	.056258
16				.000289	.019296
17				.000022	.004796
18				.000001	.000852
19					.000106
20					.000009
21					.000001

Table A.4. Exact Null Distribution of S^+_{RSS}

$P(S^+_{RSS} = y)$

y	k=5,n=2	k=5,n=3	k=5,n=4	k=5,n=5	k=5,n=6
0	.000005				
1	.000389	.000001			
2	.009028	.000056			
3	.071664	.001090	.000008		
4	.240967	.010496	.000140	.000001	
5	.355893	.054539	.001485	.000019	
6	.240967	.160564	.009766	.000209	.000003
7	.071664	.273253	.041152	.001591	.000029
8	.009028	.273253	.113481	.008397	.000248
9	.000389	.160564	.207355	.031225	.001526
10	.000005	.054539	.253225	.082821	.006972
11		.010496	.207355	.157946	.023864
12		.001090	.113481	.217791	.061679
13		.000056	.041152	.217791	.121046
14		.000001	.009766	.157946	.181099
15			.001485	. 082821	.207068
16			.000140	.031225	.181099
17			.000008	.008397	.121046
18				.001591	.061679
19				.000209	.023864
20				.000019	.006972
21				.000001	.001526
22					.000248
23					.000029
24					.000003

ACKNOWLEDGEMENTS

We thank the two referees and the editor for helpful comments.

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Received May, 1994; Revised January, 1996.