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ON A CONJECTURE BY ERDŐS AND ITS EXTENSION TO ADDITIVE FUNCTIONS ON THE SET OF PAIRS OF INTEGERS

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ABSTRACT

It was conjectured by P. Erdős in 1947, that if f is a real-valued additive arithmetic function and J is a bounded interval such that $\{m\colon f(m) \in J\}$ has positive density, then f has a distribution. It was shown to be true under some additional assumptions, in late sixties and seventies. An extension of this to arithmetic functions on the set of pairs of positive integers is partially solved in this paper. The natural density on a set of pairs of positive integers can be defined in more than one way. The solution to the problem seems to depend on the particular density considered.

1. INTRODUCTION

Let D be a set of natural numbers. If

$$\gamma(D) = \lim_{n \to \infty} \frac{1}{n} \#\{1 \leqslant m \leqslant n\colon m \in D\}$$

exists, then $\gamma(D)$ is called the natural density D. A real-valued arithmetic function f is called additive if

$$f(mn) = f(m) + f(n),$$

whenever $(m, n) = 1$. A real-valued arithmetic function f is said to have a distribution if there exists a probability distribution function F such that for all its continuity points c, $\gamma(m\colon f(m) \leqslant c)$ exists and equals $F(c)$.

Erdős (1947) conjectured that, if the natural density of $\{m\colon f(m) \in J\}$ exists and is positive for some bounded interval J, then f has a distribution. The

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conjecture was partially solved by Paul (1967) and by Babu (1977). See also Chapter 4 of (Babu, 1978) for details. In particular the distribution of f exists if one of the following holds:

i) $f \geqslant 0$, $\limsup_{n\to\infty} \frac{1}{n}\#\{1 \leqslant m \leqslant n\colon f(m) \leqslant c\} > 0$ for some $c > 0$,
ii) $\limsup_{n\to\infty} \frac{1}{n}\#\{1 \leqslant m \leqslant n\colon f(m) = c\} > 0$ for some real c,
iii) $\gamma(m\colon f(m) \in J) = 1$ for some bounded interval J.

There is a close relation between the conjecture and the properties of the independent Bernoulli random variables $\{X_p\colon p\ prime\}$ satisfying

$$P(X_p = 1) = 1 - P(X_p = 0) = \frac{1}{p}\,.$$

Elliott (1980, pp. 330–331) asserts that Erdős conjecture is valid if and only if, $\sum_p f(p)X_p$ converges almost surely whenever the limit of $P(\sum_{p\leqslant n} f(p)X_p \in J)$ as $n \to \infty$ exists and is non-zero for some bounded interval J. Here and in what follows $\sum_p$ denotes sum over prime numbers p.

The case i) was extended to non-negative arithmetic functions on the set of pairs of natural numbers $\mathbf{Z}_2$ by Babu (1976a). That proof makes full use of non-negativity of the function to reduce it to the problem for additive functions on the natural numbers. In this paper the conjecture by Erdős is set in the context of arithmetic functions on $\mathbf{Z}_2$, and is established under some minor additional assumptions.

2. ARITHMETIC FUNCTIONS ON $\mathbf{Z}_2$

Let E be subset of the set of pairs of natural numbers $\mathbf{Z}_2$, and let

$$\nu_{x,y}(E) = \frac{1}{xy}\,\#\{[m,n] \in E\colon\ 1 \leqslant m \leqslant x; 1 \leqslant n \leqslant y\}.$$

H. Delange (1969) defined the density E in two different ways. If $\nu_{x,y}(E)$ tend to a limit $\delta(E)$ as x and y tend to infinity independently, then $\delta(E)$ is called the natural density of E. For some E, this limit may not exist, but a weaker version exists. Instead of letting x and y tend to infinity independently, one can let x and y approach infinity such that $y/x \to \lambda > 0$. If this weaker limit exists, it is denoted by $\delta_\lambda(E)$. If $\delta(E)$ exists, then obviously $\delta_\lambda(E)$ exists and $\delta(E) = \delta_\lambda(E)$, for all $\lambda > 0$. The converse is not true.

A real-valued arithmetic function h on $\mathbf{Z}_2$ is called additive if

$$h(m_1m_2, n_1n_2) = h(m_1, n_1) + h(m_2, n_2),$$

whenever $(m_1n_1, m_2n_2) = 1$. An additive function on $\mathbf{Z}_2$ is said to have a distribution, with respect to the density δ, if there exists a probability distribution function F such that $\delta(\{[m,n] \in \mathbf{Z}_2\colon\ h(m,n) \leqslant c\})$ exists and equals $F(c)$, for all continuity points c of F. Distribution with respect to δ_λ is defined similarly. A result similar to Erdős–Wintner theorem holds in the case of the density δ. Babu (1976a, Theorem 2) has shown that a real-valued additive arithmetic function h has distribution with respect to the density δ, if and only if the series

$$\sum_p \frac{1}{p} h^*(p,1), \qquad \sum_p \frac{1}{p} h^*(1,p) \tag{1}$$

and

$$\sum_p \frac{1}{p} \left((h^*(p,1))^2 + (h^*(1,p))^2\right) \tag{2}$$

converge, where $z^* = z$ or 1 according as $|z| \leqslant 1$ or not.

The situation is different if the density δ_λ is used. The additive function ℓ defined by

$$\ell(m,n) = \log m - \log n,$$

does not have a distribution with respect to δ. But the distribution L_λ with respect to δ_λ exists for every $\lambda > 0$. It is interesting to note that L_λ is different for different λ. It is easy to show that for any real c,

$$L_\lambda(c) = L(c + \log \lambda), \tag{3}$$

where L is the double exponential distribution given by

$$L(c) = \begin{cases} \frac{1}{2}e^c & \text{if } c \leqslant 0, \\ 1 - \frac{1}{2}e^{-c} & \text{if } c > 0. \end{cases} \tag{4}$$

In particular, as $x \longrightarrow \infty$,

$$\nu_{x,x}\{[m,n]\colon\ |\ell(m,n)| \leqslant 1\} \longrightarrow 1 - e^{-1} > 0. \tag{5}$$

On the other hand, as $x \longrightarrow \infty$,

$$\nu_{x,x^3}\{[m,n]\colon\ |\ell(m,n)| \leqslant 1\} \longrightarrow 0. \tag{6}$$

So ℓ does not have a distribution with respect to δ. Thus the density δ_λ has the ability to handle additional logarithmic factor. We treat the problem of the existence of distributions with respect to the two densities separately.

3. MAIN RESULTS

For the rest of the paper, let μ denote a finite non-null measure on the interval $(a,b]$, which is not uniform on $(a,b]$. That is $\mu(c,d]$ is not a function of the length $d-c$ of the interval for all sub-intervals $(c,d] \subset (a,b]$. Let h denote a real-valued additive arithmetic function on $\mathbb{Z}_2$. We now state the main results on the existence of a distribution of the values of h.

THEOREM 1. *Suppose* $\delta(\{[m,n]\colon c < h(m,n) \leqslant d\})$ *exists and equals* $\mu(c,d]$ *for all* $(c,d] \subset (a,b]$. *Then h has a distribution with respect to the density* δ.

In this case, the series (1) and (2) converge. The situation is different in the case of δ_λ.

THEOREM 2. *Suppose* $\delta_\lambda(\{[m,n]\colon c < h(m,n) \leqslant d\})$ *exists and equals* $\mu(c,d]$ *for all* $(c,d] \subset (a,b]$. *Then* $h(m,n) = r(\log m - \log n) + g(m,n)$ *for some real number r, where g is an additive function for which the series*

$$\sum_p \frac{1}{p}\left(g^*(p,1) + g^*(1,p)\right) \quad \text{and} \quad \sum_p \frac{1}{p}\left((g^*(p,1))^2 + (g^*(1,p))^2\right)$$

converge. Further both g and h have distributions with respect to the density δ_λ. *The distribution G of g does not depend on λ, and the distribution H_λ of h is given by*

$$H_\lambda(c) = \int G(c - rv) L_\lambda(v)\,\mathrm{d}v, \tag{7}$$

where L_λ is given in (3).

Remark. Even if $r = 0$, h may not have a distribution with respect to the density δ. It will be clear from the proofs that a logarithmic type density is more appropriate in the consideration of distribution of values of additive arithmetic functions on natural numbers as well as on $\mathbb{Z}_2$ than δ, or δ_λ. See equation (9) below.

We need the following preliminary lemmas, which are of interest on their own. The first lemma is from the probability theory.

LEMMA 1. *Let $\{Y_n\}$ be a sequence of independent random variables. If*

$$\limsup_{n\to\infty} P\left(\left|\sum_{i=1}^n Y_i\right| \leqslant M\right) > 0, \qquad \text{for some } M > 0,$$

then there exists a sequence $\{d_n\}$ of real numbers such that $\sum_{n=1}^\infty (Y_n + d_n)$ *converges almost everywhere.*

For a proof see (Doob, 1953, p. 121).

LEMMA 2. *If for some* $M > 0$,

$$\liminf_{x,y\to\infty} \nu_{x,y}([m,n]\colon\ |h(m,n)| \leqslant M) > 0 \tag{8}$$

then the partial sums

$$A(h;x,y) = \sum_{p\leqslant x} \frac{1}{p}\, h^*(p,1) + \sum_{p\leqslant y} \frac{1}{p}\, h^*(1,p)$$

are bounded, and

$$\sum_{p} \frac{1}{p} \left((h^*(p,1))^2 + (h^*(1,p))^2\right) < \infty.$$

In view of (5) and (6), the lim inf in (8) cannot be replaced by lim sup.

Proof. We first introduce some related random variables. Let $\{U_p\colon \textit{p prime}\}$ be a sequence of independent geometric random variables, i.e.

$$P(U_p = k) = p^{-k}\Big(1 - \frac{1}{p}\Big), \qquad k = 0, 1, 2, \ldots.$$

Let $\{V_p\}$ be an independent copy of the sequence $\{U_p\}$ defined on the same probability space. Let

$$W_p = h(p^{U_p}, p^{V_p}).$$

Clearly $\{W_p\colon \textit{p prime}\}$ are independent random variables. For any set S of real numbers, let $I(h;S;m,n) = 1$, or $= 0$ according as $h(m,n) \in S$ or not. It is not difficult to conclude that for any Borel set S on the line

$$P\Big(\sum_{p\leqslant x} W_p \in S\Big) = \prod_{p\leqslant x} \Big(1 - \frac{1}{p}\Big)^2 \sum{}^{x} \frac{1}{mn}\, I(h;S;m,n), \tag{9}$$

where $\sum^x$ denotes the sum over all integers m, n which are not divisible by any prime number p larger than x. Further observe that, for any $E \subset \mathbf{Z}_2$,

$$\begin{aligned}\int_1^x\!\!\int_1^y \frac{1}{uv}\, \nu_{u,v}(E)\, du\, dv &= \sum_{m\leqslant x;n\leqslant y} I_E(m,n)\Big(\frac{1}{m} - \frac{1}{x}\Big)\Big(\frac{1}{n} - \frac{1}{y}\Big)\\ &\leqslant \sum_{m\leqslant x;n\leqslant y} \frac{1}{mn}\, I_E(m,n),\end{aligned}$$

where I_E denotes the indicator function of the set E. Consequently, if (8) holds, then

$$\liminf_{x,y\to\infty} \frac{1}{(\log x)(\log y)} \sum_{m\leqslant x;n\leqslant y} \frac{1}{mn}\, I(h;[-M,M];m,n) > 0.$$

This implies

$$\limsup_{x\to\infty}\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^2\sum{}^x\frac{1}{mn}I(h;[-M,M];m,n)>0. \tag{10}$$

By (9) and (10), we have

$$\limsup_{x\to\infty}P\left(\left|\sum_{p\leqslant x}W_p\right|\leqslant M\right)>0.$$

By Lemma 1, there exists a sequence of real numbers $\{d_p\}$ such that

$$\sum_p(W_p+d_p)\qquad\text{converges a.e.} \tag{11}$$

We shall now show that $d_p\to0$ as $p\to\infty$. Since $W_p+d_p\to0$ a.e., for any $\varepsilon>0$, there exists $k\geqslant5$, such that for $p>k$,

$$\begin{aligned}\frac{1}{2}&\leqslant P(|W_p+d_p|<\varepsilon)\\&\leqslant P(|W_p+d_p|<\varepsilon,W_p=0)+P(|W_p+d_p|<\varepsilon,W_p\neq0)\\&\leqslant P(|W_p+d_p|<\varepsilon,W_p=0)+\frac{2}{p}.\end{aligned}$$

This leads to a contradiction unless $|d_p|<\varepsilon$ for $p>k$. So $d_p\to0$. As $\sum_p(W_p+d_p)$ converges, we have by Kolmogorov's three series theorem that $\sum_p P(|W_p+d_p|>1/2)<\infty$. This implies, since $d_p\to0$, that $\sum_p P(|W_p|>1/2)<\infty$. Consequently

$$\sum_{|h(p,1)|\geqslant1}\frac{1}{p}+\sum_{|h(1,p)|\geqslant1}\frac{1}{p}<\infty.$$

Another application of Kolmogorov's three series theorem gives the convergence of (2), and that (11) holds with $d_p=-\frac{1}{p}(h^*(p,1)+h^*(1,p))$. Let F denote the distribution of $\sum_p(W_p-\frac{1}{p}(h^*(p,1)+h^*(1,p)))$. Now the standard arguments using Turán–Kubilius type inequality (see (Kubilius, 1964)) leads to

$$\nu_{x,y}([m,n]:\ h(m,n)-A(h;x,y)\leqslant c)\longrightarrow F(c), \tag{12}$$

as $x,y\to\infty$, for all continuity points c of F. Boundedness of the partial sums $\{A(h;x,y)\}$ follows from (8) and (12).

Proof of Theorem 1. In view of (12), it is enough to prove that $A(h;x,y)$ tends to a limit as x and y tend to infinity. Let θ_1 and θ_2 denote lower and upper limits of the bounded partial sums $\{A(h;x,y)\colon\ x,y\}$. By (12), it follows that $\mu[c,d] = F(d+\theta) - F(c+\theta)$, for all $[c,d] \subset [a,b]$ and for all $\theta \in [\theta_1,\theta_2]$. This clearly violates the assumed non-uniformity of the measure μ, if $\theta_1 \neq \theta_2$. So $A(h;x,y)$ tends to a limit as x and y tend to infinity, completing the proof of Theorem 1.

Lemma 2 is not suitable for proving Theorem 2, as it involves additional logarithmic factor. We require the following lemma.

LEMMA 3. *Suppose for some* $M > 0$ *and,* $\beta > 0$,

$$\liminf_{x\to\infty} \nu_{x,x}([m,n]\colon\ |h(m,n)| \leqslant M) > \beta > 0. \tag{13}$$

Then for some real number r,

$$h(m,n) = r(\log m - \log n) + g(m,n),$$

where g *is an additive function for which the sequence* $\{A(g;x,x)\colon\ x\}$ *is bounded and*

$$\sum_p \frac{1}{p}\left((g^*(p,1))^2 + (g^*(1,p))^2\right) < \infty. \tag{14}$$

In addition, there exists a probability distribution function F *such that*

$$\nu_{x,y}([m,n]\colon\ g(m,n) - A(g;m,n) \leqslant c) \longrightarrow F(c), \tag{15}$$

for all continuity points c *of* F.

Proof. We first establish (14) and (15). We use the ideas due to Erdős, see the proof of Theorem 7.2 on page 258 of (Elliott, 1980). Let

$$\psi(x,t) = \frac{1}{x^2} \sum_{1\leqslant m,n\leqslant x} e^{ith(m,n)}.$$

By a result of Delange (1970), which is stated as Lemma 3 in (Babu, 1976b), the limit $\xi(t)$ of $|\psi(x,t)|$ exists, as $x \to \infty$. Following the proof of Theorem 7.2 of Elliott (1979), we get for all $\eta > 4M$, that

$$\limsup_{x\to\infty} \left(\nu_{x,x}([m,n]\colon\ |h(m,n)| \leqslant M)\right)^2 \leqslant 2\int \eta(\xi(t))^2 \left(\frac{\sin \eta t\pi}{\eta t\pi}\right)^2 dt.$$

This and (13) imply that the Lebesgue measure of $B = \{t\colon\ \xi(t) > 0\}$ is non-zero. Which in turn implies, as in the proof of Theorem 7.2 of Elliott (1979), that for some real numbers q, r, and an additive function g satisfying (14),

$$h(m,n) = r\log m + q\log n + g(m,n).$$

Now as in the proof of Lemma 2, the standard arguments using Turán–Kubilius type inequality lead to the existence of a probability distribution function F satisfying,

$$\nu_{x,y}([m,n]\colon\ g(m,n) - A(g;x,y) \leqslant c) \longrightarrow F(c), \tag{16}$$

as $x, y \to \infty$, for all continuity points c of F. Now (15) follows, since for any $\varepsilon > 0$,

$$\sup_{\varepsilon<\varepsilon_1<\varepsilon_2\leqslant 1} |A(g;x,y) - A(g;\varepsilon_1 x, \varepsilon_2 y)| \leqslant \sum_{\varepsilon x<p\leqslant x} \frac{1}{p} + \sum_{\varepsilon y<p\leqslant y} \frac{1}{p} \longrightarrow 0, \tag{17}$$

as $x, y \to \infty$.

To show $q = -r$, note that

$$(A(g;x,x))^2 \leqslant 2\left(\sum_{p\leqslant x} \frac{1}{p}\left((g^*(p,1))^2 + (g^*(1,p))^2\right)\right)\sum_{p\leqslant x}\frac{1}{p}.$$

Hence we have by (14) and (16), that for some real number $K > 1$

$$\nu_{x,x}([m,n]\colon\ |g(m,n)| \leqslant K \log\log x) \longrightarrow 1, \tag{18}$$

as $x \to \infty$. As $h(m,n) - g(m,n) = r\log m + q\log n$, we have by (13) and (18) that

$$\begin{aligned} 0 &< \limsup_{x\to\infty} \nu_{x,x}([m,n]\colon\ |\log(m^r n^q)| \leqslant K\log\log x) \\ &= \limsup_{x\to\infty} \nu_{x,x}\left([m,n]\colon\ \frac{x}{\log x} \leqslant m, n \leqslant x,\right. \\ &\qquad\qquad \left.\frac{1}{(\log x)^K} \leqslant m^r n^q \leqslant (\log x)^K\right). \end{aligned} \tag{19}$$

Note that if $x(\log x)^{-1} \leqslant m,\ n \leqslant x$, then

$$(\log x)^{-|r|-|q|} \leqslant (m/x)^r (n/x)^q \leqslant (\log x)^{|r|+|q|}.$$

So by (19) we have that $r = -q$. Hence $h(m,n) = r(\log m - \log n) + g(m,n)$

Finally, to establish boundedness of the sequence $\{A(g;x,x)\}$, note that by (4) and (13), we have

$$\begin{aligned} &\liminf_{x\to\infty} \nu_{x,x}([m,n]\colon\ |g(m,n)| \leqslant M + |r|\log(4/\beta)) \\ &\quad\geqslant \liminf_{x\to\infty} \nu_{x,x}([m,n]\colon\ |h(m,n)| \leqslant M) \end{aligned}$$

$$
\begin{aligned}
&- \liminf_{x\to\infty} \nu_{x,x}([m,n]:\ |\log m - \log n| \geqslant \log(4/\beta)) \\
&\geqslant \frac{1}{2}\beta > 0.
\end{aligned}
$$

This inequality and (16) together imply that the sequence $\{A(g;x,x)\}$ is bounded. This completes the proof of Lemma 3.

Proof of Theorem 2. By (15),

$$
B(g,t;u,v) = \frac{1}{uv} \sum_{m\leqslant u;n\leqslant v} \mathrm{e}^{it(g(m,n)-A(g;m,n))} \longrightarrow B_g(t),
$$

as $u,v \to \infty$, where B_g denotes the Fourier transform of the distribution function F appearing in (15). Summing by parts, we obtain

$$
\begin{aligned}
&\frac{1}{xy} \sum_{m\leqslant x;n\leqslant y} \mathrm{e}^{it(f(m,n)-A(g;m,n))} \\
&= \frac{r^2t^2}{xy} \int_1^x \int_1^y u^{itr} v^{-itr} B(g,t;u,v)\,\mathrm{d}v\,\mathrm{d}u \\
&\quad - irty^{-irt} \frac{1}{x} \int_1^x u^{itr} B(g,t;u,y)\,\mathrm{d}u \\
&\quad + irtx^{irt} \frac{1}{y} \int_1^y v^{-itr} B(g,t;x,v)\,\mathrm{d}v + x^{irt} y^{-irt} B(g,t;x,y) \\
&\longrightarrow \frac{\lambda^{-itr}}{1+t^2r^2} B_g(t),
\end{aligned}
$$

as $x,y \to \infty$ such that $y/x \to \lambda > 0$. Hence it follows by (16) and (17), that

$$
\nu_{x,y}([m,n]:\ h(m,n) - A(g;x,y) \leqslant c) \longrightarrow \int F(c-rs)\,\mathrm{d}L_\lambda(s), \tag{20}
$$

as $x,y \to \infty$ such that $y/x \to \lambda > 0$. Now as in the proof of Theorem 1, it follows that $A(g;x,\lambda x)$ tends to a limit. Consequently, g and h both have distributions G and H_λ, and hence (7) follows by (20). This completes the proof.

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