Limit theorems for random permutations

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Let $\mathbf{S}_n$ denote the symmetric group of permutations on $\{1, \ldots, n\}$. In 1967, P. Erdős and P. Turán established the weak convergence result,

$$\frac{1}{n!}\#\left(\sigma \in \mathbf{S}_n : \log Ord(\sigma) - 0.5\log^2 n \le \frac{y}{\sqrt{3}}\log^{3/2} n\right) \to \Phi(y), \tag{1}$$

as $n \to \infty$, where Φ denotes the standard normal distribution function, and $Ord(\sigma)$ denotes the order of a permutation σ. As $Ord(\sigma)$ depends only on the conjugate class containing σ, Ord can be treated as a function on the space $\tilde{\mathbf{S}}_n$ of conjugate classes of $\mathbf{S}_n$. The space $\tilde{\mathbf{S}}_n$ can be identified by the set of vectors $\bar{k} = (k_1, \ldots, k_n)$, of non-negative integers representing partitions of n.

A general family of measures $\nu_{n,\theta}, \theta > 0$, on $\tilde{\mathbf{S}}_n$ were described by Ewens (1972) in connection with models in population genetics. The measure $\nu_{n,\theta}$ on $\tilde{\mathbf{S}}_n$, known as the Ewens sampling formula is given by

$$\nu_{n,\theta}\{(k_1, \ldots, k_n)\} = \frac{n!}{\theta(\theta+1)\cdots(\theta+n-1)}\prod_{j=1}^{n}\left(\frac{\theta}{j}\right)^{k_j}\frac{1}{k_j!}, \tag{2}$$

where $\theta > 0, k_j \ge 0$ and $1k_1 + \cdots + nk_n = n$. The measure induced on $\tilde{\mathbf{S}}_n$ by the uniform measure on $\mathbf{S}_n$ considered in (1) is $\nu_{n,\theta}$ with $\theta = 1$. Mixtures of $\nu_{n,\theta}$ also have applications in Bayesian statistics. It is well known that

$$\nu_{n,\theta}(\bar{k}) = P(\xi_1 = k_1, \ldots, \xi_n = k_n \,|1\xi_1 + \cdots + n\xi_n = n), \tag{3}$$

where ξ_j, $1 \le j \le n$ are independent Poisson random variables satisfying $\mathbf{E}\xi_j = \theta/j$.

In the last few decades several authors contributed to limit theorems for some specific functions on $\mathbf{S}_n$ such as $w(\sigma)$ representing the total number of cycles of σ. However, very few results on the necessity part are known in the literature.

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Recent results of Babu and Manstavičius on functional limit theorems in $\mathbf{C}[0,1]$ and in $\mathbf{D}[0,1]$, for partial sum processes based on the measures $\nu_{n,\theta}$ for general additive functions on $\hat{\mathbf{S}}_n$ are presented here. The results are established using methods from probabilistic number theory. Let h be an additive function on $\mathbf{S}_n$, given by

$$h(\sigma) = \sum_{j=1}^{n} h_j(k_j(\sigma)), \tag{4}$$

for each $\sigma \in \mathbf{S}_n$, where $h_j(0) = 0$, $h_j(k)$, $j \geq 1$, $k \geq 0$, is some double sequence of real numbers, and $k_j(\sigma)$ denotes the number of cycles of σ of length j. Define

$$A(u,n) := \sum_{j\leq u} \frac{1}{j} h_j(1)\theta, \qquad B^2(u) := \sum_{j\leq u} \frac{1}{j} h_j(1)^2\theta,$$

$$y(t) := y_n(t) = \max\{u \leq n : B^2(u) \leq tB^2(n)\}, \quad t \in [0,1], \text{ and}$$

$$H_{n,h} := H_{n,h}(\sigma,t) = \frac{1}{B(n)}\Big(\sum_{j\leq y(t)} h_j(k_j(\sigma)) - A(y(t),n)\Big) \quad t \in [0,1].$$

We consider the weak convergence of the process $H_{n,h}$ to the Wiener measure W.

Theorem 1. *Suppose h is a real additive function on $\mathbf{S}_n$ given by (4) and $B(n) \to \infty$. The sequence of measures $\nu_{n,\theta} \cdot H_{n,h}^{-1}$ converges weakly to the Wiener measure W if and only if, for each $\varepsilon > 0$,*

$$\Lambda_n(\varepsilon) := \frac{1}{B^2(n)} \sum_{\substack{j=1 \\ |h_j(1)|\geq \varepsilon B(n)}}^{n} \frac{1}{j} h_j(1)^2 = o(1). \tag{5}$$

It is interesting to note that a Lindeberg type condition (5) is necessary for the dependent process to converge to the Brownian Motion, while it is not the case for the convergence of the one dimensional distributions. The following example illustrates this phenomenon when $\theta = 1$. Let γ_j denote the fractional part of $j\sqrt{2}$ and let the additive function h is given by (4) and

$$h_j(k) = \begin{cases} k\sqrt{j}\,\Phi^{-1}(\gamma_j) & \text{if } |\Phi^{-1}(\gamma_j)| \leq \log j, \\ 0 & \text{otherwise.} \end{cases}$$

Then it can be shown that condition (5) is violated but the one dimensional limiting distribution of $(h(\cdot) - A(n,n))/B(n)$ is the standard normal distribution Φ.

By changing the scaling factor, Babu and Manstavičius have shown under very general conditions that a partial sum process converges weakly in a function space to a stable process if and only if the corresponding process defined through sums of independent random variables converges weakly. As a consequence of this result, necessary and sufficient conditions for weak convergence to a stable process are derived.

To state the results, let the normalizing factor $\beta(n) > 0$ satisfy $\beta(n) \to \infty$. The sequence $\{\beta(n)\}$ need not be monotone. Define

$$C(u,n) \quad = \sum_{j\le u} \left(\frac{h_j(1)}{\beta(n)}\right)^{*2} \frac{1}{j}, \qquad A(u,n,\beta) = \theta \sum_{j\le u} \left(\frac{h_j(1)}{\beta(n)}\right)^{*} \frac{1}{j}$$

$$s(t) := s_n(t,C) = \max\{l \le n : C(l,n) \le tC(n,n)\}, \quad t \in [0,1], \text{ and}$$

$$R_{n,h} := R_{n,h}(\sigma,t) = \frac{1}{\beta(n)} \sum_{j\le s(t)} h_j(k_j(\sigma)) - A(s(t),n,\beta), \quad t \in [0,1].$$

We shall consider the weak convergence (denoted by $\Rightarrow$) of the process R_n under the measure $\nu_{n,\theta}$, in the space $\mathbf{D}[0,1]$ endowed with the Skorohod topology. The corresponding process $X_{n,h}$ with independent increments is defined by

$$X_{n,h} := X_{n,h}(t) = \frac{1}{\beta(n)} \sum_{j\le s(t)} h_j(1)\xi_j - A(s(t),n,\beta), \quad t \in [0,1].$$

Theorem 2. *Suppose X is a random element in $\mathbf{D}[0,1]$ for which the distribution of $X(1)$ is non-degenerate. Further assume that for some $0 < \eta < 1$ and for all but countably many $t \in (\eta,1)$, the distribution of $X(1) - X(t)$ is absolutely continuous with respect to the Lebesgue measure on the real line. Then $R_{n,h} \Rightarrow X$ if and only if $X_{n,h} \Rightarrow X$.*

When $\theta = 1$, as in the context of weak convergence to a Gaussian law in the one-dimensional case, the following counter example shows that $X_{n,h}(1)$ and $R_{n,h}(1)$ need not have the same limiting distribution.

Counter Example. Let $0 < \alpha < 2$. Let F denote the distribution function of the stable law with characteristic function ϕ_α given by, $\phi_\alpha(s) = e^{-|s|^\alpha}$. Define

$$h_j(1) = \begin{cases} j^{1/\alpha} F^{-1}(\gamma_j) & \text{if} \quad |F^{-1}(\gamma_j)| \le j^{1/\alpha} \\ 0 & \text{otherwise.} \end{cases}$$

If $\beta(n) = n^{1/\alpha}$, then for the completely additive function $h(\sigma) = \sum_{j=1}^{n} h_j(1)k_j(\sigma)$, the distribution of $(h(\cdot)/\beta(n)) - A(n,n,\beta)$ converges weakly to F. However, $X_{n,h}(1)$ converges weakly to a distribution with characteristic function ϕ given by

$$\phi(s) = \exp\left\{\int_0^1 \frac{1}{y}\left(e^{-y|s|^\alpha} - 1\right)dy\right\}.$$

The assumption on $X(1) - X(t)$ in Theorem 2 can be relaxed by controlling the growth of $\beta(n)$. An important aspect of the proof of Theorem 2 involves in showing that β is a slowly varying function. We have

Theorem 3. *Suppose X is a stochastic process with independent increments having paths in $\mathbf{D}[0,1]$ and such that $P(X(1) = a) < 1$ for any real a. Then $R_{n,h} \Rightarrow X$ if and only if $X_{n,h} \Rightarrow X$ and β is slowly varying.*