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Probabilistic number theory and random permutations: Functional limit theory

Gutti Jogesh Babu

Dedicated to Professor K. Ramachandra on his 70th birthday

Abstract

The ideas from Probabilistic Number Theory are useful in the study of measures on partitions of integers. Connection between the Ewens sampling formula in population genetics and the partitions of an integer generated by random permutations will be discussed. Functional limit theory for partial sum processes induced by Ewens sampling formula is reviewed. The results on limit processes with dependent increments are illustrated.

1 Introduction

In the last few decades, mathematical population geneticists have been exploring the mechanisms that maintain diversity in a population. In 1972, Ewens established a formula to describe the probability distribution of a sample of genes from a population that has evolved over many generations, by a family of measures on the set of permutations of the first n integers (equivalently on the set of partitions of n). The Ewens formula can be used to test if the popular assumptions are consistent with data, and to estimate the parameters. The statistics that are useful in this connection will generally be expressed as functions of the sums of transforms of the 'allelic partition'. Such statistics can be viewed as functions of a process on the permutation group of integers.

In a series of papers [3]-[7], Babu and Manstavičius, have developed necessary and sufficient conditions for the weak convergence of a partial sum process based on these measures to a process with independent increments. Under very general conditions, it has been shown that a partial sum process converges weakly in a function space if and only if a related process defined through sums of independent random variables converge. In this paper, the case where the limiting processes need not be processes with independent increments is considered. Thus, under Ewens sampling formula, the limiting process of the partial sums of dependent variables differs from that of the associated process defined through the partial sums of independent random variables. The basic ideas for proofs come from probabilistic number theory and analytic number theory. Integration over Hankel contour (see Corollary 2.1 in Section II.5.2 in [17]) plays an important role.

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2 Probabilistic Number Theory

We shall start with a brief comparative analysis of the developments in probabilistic number theory and the theory of random permutations. The uniform probability measure

$$\nu_n(A) = \frac{1}{n}\#\{1 \le m \le n : m \in A\}$$

on integers, satisfies for $k, l \ge 0$,

$$\nu_n(\alpha_p(m) = k) = \frac{1}{n}\left(\left[\frac{n}{p^k}\right] - \left[\frac{n}{p^{k+1}}\right]\right) \approx \frac{1}{p^k}\left(1 - \frac{1}{p}\right), \qquad k \ge 0,$$

and

$$\begin{aligned}\nu_n(\alpha_p(m) = k, \alpha_q(m) = l) &\approx \frac{1}{p^k}\left(1 - \frac{1}{p}\right)\frac{1}{q^l}\left(1 - \frac{1}{q}\right),\\ &\approx \nu_n(\alpha_p(m) = k)\nu_n(\alpha_q(m) = l),\end{aligned}$$

where $m = \prod_p p^{\alpha_p(m)}$ is the unique representation of integer m as the product of prime powers. It follows that α_p has asymptotically geometric distribution. In addition, α_p and α_q are asymptotically independent, where p and q are distinct primes.

The Fundamental Theorem of probabilistic number theory [15] states that

$$\nu_n(\alpha_p(m) = k_p, p \le r) = P(\xi_p = k_p, p \le r) + o(1),$$

where $r \le n^\varepsilon$ for each $\varepsilon > 0$, and $\{\xi_p\}$ are independent geometric random variables

$$P(\xi_p = k) = \frac{1}{p^k}\left(1 - \frac{1}{p}\right), \qquad k \ge 0.$$

If h is an additive arithmetic function, $h(mn) = h(m) + h(n)$, $(m, n) = 1$, then h can be represented as $h(m) = \sum_p h(p^{\alpha_p(m)})$. As $\alpha_p(m) = 1$ for a prime $\sqrt{n} < p \le n$ implies $\alpha_q(m) = 0$ for all primes $q \ne p$, $\sqrt{n} < q \le n$, it follows that they are not independent even asymptotically. To establish the limiting distribution of h under ν_n, one uses the decomposition $h(m) = h_r(m) + h^r(m)$, where

$$h_r(m) = \sum_{p \le r} h(p^{\alpha_p(m)}), \quad h^r(m) = \sum_{p > r} h(p^{\alpha_p(m)}).$$

Then the fundamental theorem of probabilistic number theory is used to approximate $\nu_n(h_r(m) \le x)$ by $P\left(\sum_{p \le r} f_p(\xi_p) \le x\right)$ and showing that the contribution of h^r is negligible, where $f_p(k) = h(p^k)$.

Similar ideas are used in obtaining functional limit theorems by Babu [2] for the partial sum process

$$X_n(t) = \left(1/\sqrt{B(n,n)}\right)\sum{}' h(q), \quad B(n,k) = \sum_{p \le k} \frac{1}{p} h^2(p), \quad t \in [0,1],$$

where the sum $\sum'$ is taken over all primes $q \le n$ satisfying $B(n, q) \le tB(n, n)$.

3 Statistical Group Theory

Similar approach can be used in the study of statistical group theory and in particular random permutations. Let $\mathbf{S}_n$ denote the group of permutations on $\{1, \ldots, n\}$ Each permutation σ can be decomposed as $\sigma = \kappa_1 \ldots \kappa_\omega$ where $\omega(\sigma)$ denotes the number of cycles of σ, and κ_i denote the independent cycles. For example, the permutation τ that maps $\{1, 2, 3, 4, 5, 6, 7, 8\}$ to $\{5, 3, 6, 1, 8, 2, 7, 4\}$ has three cycles (1 5 8 4), (2 3 6), (7) and can be represented as $\tau = (1\ 5\ 8\ 4)\ (2\ 3\ 6)\ (7)$. Thus $Ord(\tau) = 12$, where the order $Ord(\sigma)$ of permutation σ is defined to be the smallest k such that $\sigma^k =$ identity permutation. If $k_j(\sigma)$ denotes the number of cycles of length j of σ, then $\omega(\sigma) = k_1(\sigma) + \cdots + k_n(\sigma)$ and $Ord(\sigma) = \text{l.c.m.}\{j \le n : k_j(\sigma) > 0\}$.

Goncharov, Erdos-Turan, and others contributed to the theory. In 1942, V. L. Goncharov [12] has shown that

$$\frac{1}{n!}\,\#\left\{\sigma \in \mathbf{S}_n : \omega(\sigma) - \log n < x\sqrt{\log n}\right\} \to \Phi(x),$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{1}{2}u^2}du$. In 1965, Erdös and Turán [10] have established that

$$\frac{1}{n!}\,\#\left\{\sigma \in \mathbf{S}_n : \log Ord(\sigma) - \frac{1}{2}\log^2 n \le \frac{x}{\sqrt{3}}\log^{3/2} n\right\} \to \Phi(x).$$

However, in these and other early works, there is no trace of the use of ideas from probabilistic number theory, though the functions are similar.

The equivalent relation, $\sigma \sim \tau$ if $k_j(\sigma) = k_j(\tau)$ for all j, partitions $\mathbf{S}_n$ into equivalence classes, known as conjugate classes. Hence we can identify σ with the vector $\underline{k} = (k_1(\sigma), \ldots, k_n(\sigma))$, where $1k_1(\sigma) + \cdots + nk_n(\sigma) = n$. This leads to random partitions of integer n.

4 Ewens Sampling Formula

The family of probability measures on the symmetric group $\mathbf{S}_n$ of permutations on $\{1, \ldots, n\}$, induced by the Ewens sampling formula (see [11]) are given by

$$\nu_{n,\theta}(\underline{k}) := \frac{n!}{\theta_{(n)}}\prod_{j=1}^{n}\left(\frac{\theta}{j}\right)^{k_j}\frac{1}{k_j!}, \quad \underline{k} := (k_1, \ldots, k_n) \in \mathbf{Z}^{+^n},$$

for the partition $n = 1k_1 + \cdots + nk_n$, $\underline{n} \in \mathbf{N}$, and 0 otherwise, where $\theta > 0$, and $\theta_{(n)} = \theta(\theta+1)\cdots(\theta+n-1)$. The quantity $\nu_{n,\theta}(\underline{k})$ can also be viewed as the probability measure on the class of conjugate elements $\sigma \in \mathbf{S}_n$, all having $k_j(\sigma) = k_j$ cycles of length j, $1 \le j \le n$. The probability measure $\nu_{n,\theta}$ is induced by the measure $\nu'_{n,\theta}$ on $\mathbf{S}_n$, that assigns a mass proportional to $\theta^{w(\sigma)}$ for $\sigma \in \mathbf{S}_n$, where $w(\sigma) = k_1(\sigma) + \cdots + k_n(\sigma)$ denotes the total number of cycles of σ. This can be seen from

$$\nu'_\theta(\sigma) = \theta^{w(\sigma)}\left(\sum_{\tau \in \mathbf{S}_n}\theta^{w(\tau)}\right)^{-1} = \frac{\theta^{w(\sigma)}}{\theta_{(n)}}.$$

Thus, we use this probability measure on $\mathbf{S}_n$ and leave the same notation $\nu_{n,\theta}$ for it.

The case $\theta = 1$ corresponds to the measure induced by the uniform probability $(1/n!)\,\#\{\sigma \in \mathbf{S}_n : \cdots\}$ on $\mathbf{S}_n$. If $k_j(\sigma) = 1$ for some $\frac{n}{2} < j \leq n$, then $k_i(\sigma) = 0$ for all $\frac{n}{2} < i \leq n$, $i \neq j$.

As mentioned in the introduction §1, the Ewens formula describes the probability law of a sample of n genes from a population that has evolved over many generations. The domain of $\nu_{n,\theta}$, $\{(k_1, \ldots, k_n) : 1k_1 + \cdots + nk_n = n\}$ is same as that of the Allelic Partition $\bar{k} = (k_1, \ldots, k_n)$, where k_j denotes the number of alleles appearing j times. The distribution of *Allelic Partition* has all the information available in the sample of n genes. Hence, the Ewens formula can be used to test if the popular assumptions are consistent with data, and to estimate the parameters.

This motivates consideration of additive functions on $\mathbf{S}_n$. A function $h : \mathbf{S}_n \to \mathbf{R}$ is called additive if $h_j(0) = 0$ and $h(\sigma) = \sum_{j=1}^{n} h_j(k_j(\sigma))$, $\sigma \in \bar{k}$. Kolchin and Chistyakov [14] showed that $\nu_{n,1}(\sum_{j\leq r} a_{jn} k_j(\sigma) - A_r < x)$ converges for some sequence $\{A_r\}$ if and only if $P(\sum_{j\leq r} a_{jn} Y_j - A_r < x)$ converges, where Y_j are independent Poisson random variable with mean $1/j$, $r = r(n) \to \infty$, and $r \log r = o(n)$.

To facilitate the study of the limiting distributions of additive functions on $\mathbf{S}_n$, Arratia and Tavaré [1] developed a result similar to fundamental result of Kubilius in probabilistic number theory, which states that

$$\nu_{n,1}(k_j(\sigma) = k_j, j \leq r) = P(Y_j = k_j, j \leq r) + O_\delta\left(exp\left\{-(1-\delta)\frac{n}{r}\log\frac{n}{r}\right\}\right).$$

The measure $\nu_{n,\theta}$ can be represented using independent Poisson random variables ξ_j with $\mathbf{E}(\xi_j) = \frac{\theta}{j}$, as

$$\nu_{n,\theta}(\bar{k}) = P(\xi_1 = k_1, \ldots, \xi_n = k_n \,|\zeta = n), \; \zeta = 1\xi_1 + \cdots + n\xi_n.$$

5 Functional Limit Theorems: Processes with independents

To state the functional limit theorems, let $h(\sigma) = \sum_{i=1}^{n} h_j(k_j(\sigma))$ denote an additive function on $\mathbf{S}_n$. Let

$$A(u) = \theta \sum_{j\leq u} \frac{1}{j}\, h_j(1), \; B^2(u) = \theta \sum_{j\leq u} \frac{1}{j}\, h_j(1)^2,$$

and

$$y_n(t) = \max\{u : B^2(u) \leq tB^2(n)\}.$$

The first functional limit theorem for $\theta = 1$ and $\omega(\sigma)$ was obtained by DeLaurentis and Pittel [8]. This was extended to general θ by Hansen [13] and Donnelly *et al.* [9]. The following theorem for general additive functions is from [3].

Theorem 1 (Babu and Manstavičius [3]). *Suppose* $B(n) \to \infty$, *and*

$$H_n(\sigma, t) = \frac{1}{B(n)}\left(\sum_{j\leq y_n(t)} h_j(k_j(\sigma)) - A(y_n(t))\right).$$

Then $\nu_{n,\theta} \cdot H_n^{-1} \Rightarrow W$ *if and only if for each* $\varepsilon > 0$,

$$\Lambda_n(\varepsilon) = \frac{1}{B^2(n)} \sum_{|h_j(1)|\geq\varepsilon B(n)} \frac{1}{j}\, h_j(1)^2 \to 0,$$

where W denotes the Brownian Motion on $[0, 1]$.

This result leads, via invariance principle of the probability theory, to the limiting distributions of functions of partial sums of h_j These results throw new light on the partitions of integers. For example the functional limit theorem implies,

$$\nu_{n,\theta}(h(\sigma) - A(n) \le xB(n)) \to \Phi(x),$$

$$\nu_{n,\theta}\left(\sup_{k\le n}\left(\sum_{j\le k} h_j(k_j(\sigma)) - A(k)\right) \le xB(n)\right) \to P\left(\sup_{0\le t\le 1} W(t) \le x\right)$$
$$= 2\Phi(x), \qquad x \ge 0$$
$$\nu_{n,\theta}\left(\sup_{k\le n}\left|\sum_{j\le k} h_j(k_j(\sigma)) - A(k)\right| \le xB(n)\right) \to P\left(\sup_{0\le t\le 1} |W(t)| \le x\right)$$
$$= \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} e^{-\pi^2(2k+1)^2/8x^2}.$$

By using a slowly varying function to scale h, Babu and Manstavičius obtained convergence to a stable processes and to general processes with independent increments. Let $\beta(n) \to \infty$ and $x^* = \min(|x|, 1)\,\mathrm{sign}(x)$. For an additive function h, let

$$c(u,n) = \sum_{j\le u} \frac{1}{j}\left(\frac{h_j(1)}{\beta(n)}\right)^{*^2}, \quad A(u,n,\beta) = \theta\sum_{j\le u} \frac{1}{j}\left(\frac{h_j(1)}{\beta(n)}\right)^{*},$$

$s_n(t) = \max\{l \le n : c(l,n) \le t\, c(n,n)\}$,

$$R_{n,h}(t) = \frac{1}{\beta(n)} \sum_{j\le s_n(t)} h_j(k_j(\sigma)) - A(s_n(t), n, \beta)$$

and

$$X_{n,h}(t) = \frac{1}{\beta(n)} \sum_{j\le s_n(t)} h_j(1)\xi_j - A(s_n(t), n, \beta).$$

The following result is from Babu and and Manstavičius [7].

Theorem 2. *In order that $R_{n,h} \Rightarrow X$, where X is a process with independent increments and the distribution of $X(1)$ is non-degenerate, it is necessary and sufficient that $X_{n,h} \Rightarrow X$ and $\beta(n)$ is slowly varying.*

If $X_{n,h} \Rightarrow X$, then the limiting process X is necessarily a process with independent increments and it satisfies $P(X(0) = 0) = 1$. It is interesting to note that the convergence of the process defined through the partial sums of dependent random variables is equivalent to the convergence of the process defined through the partial sums of the corresponding independent random variables. The result holds in spite of the strong dependent structure on $\{k_j(\sigma) : \frac{1}{2}n \le j \le n\}$.

Counter Example. In the one-dimensional case, $X_{n,h}(1)$ and $R_{n,h}(1)$ need not have the same limit. To see this, let $\theta = 1$, $0 < \alpha < 2$ and let F be the stable law with characteristic function $\phi_\alpha(s) = e^{-|s|^\alpha}$. Let γ_j denote the fractional part of $j\sqrt{2}$,

$$a(j) = \begin{cases} j^{1/\alpha} F^{-1}(\gamma_j) & \text{if } |F^{-1}(\gamma_j)| \le j^{1/\alpha} \\ 0 & \text{otherwise,} \end{cases}$$

$h(\sigma) = \sum_{j=1}^{n} k_j(\sigma)a(j)$, and $\beta(n) = n^{1/\alpha}$. Then $(h(\cdot)/\beta(n)) - A(n, n, \beta) \Rightarrow F$. But $X_{n,h}(1) \Rightarrow G$, where the characteristic function of G is

$$\phi(s) = \exp\left\{\int_0^1 \frac{1}{y}\Big(e^{-y|s|^\alpha} - 1\Big)dy\right\}.$$

6 Limit processes with dependent increments

The example above illustrates that if β is not slowly varying function, the limit process may have dependent increments. To study such limits, a generalization of the Main Lemma in [16] that involves integration over Hankel contour of the type given in Figure 1 is needed.

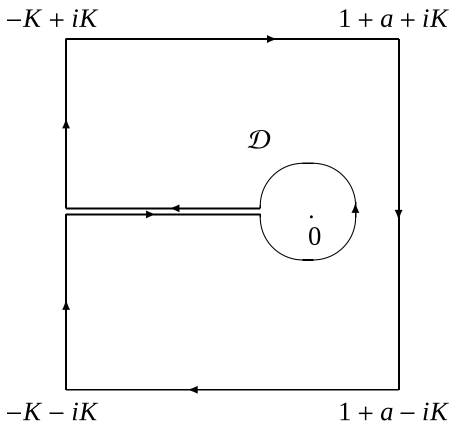

Figure 1: Contour with $a \geq 0$ and $K > 0$

However, to facilitate the discussion we consider an example with $\beta(n) = n^\rho$, $\rho > 0$. Let the additive function h on $\mathbf{S}_n$ be given by $h(\sigma) = \sum_{j=1}^{n} k_j(\sigma) j^\rho$. Let the processes H_n based on h be given by,

$$H_n(t) := H_n(t, \sigma) = (1/\beta(n)) \sum_{j \leq nt} k_j(\sigma) j^\rho = \sum_{j \leq nt} k_j(\sigma)(j/n)^\rho, \qquad 0 \leq t \leq 1.$$

Note that $H_n(1, \sigma) = 1$ for all $\sigma \in \mathbf{S}_n$ if $\rho = 1$. We now present preliminary notation and results needed in illustrating the limiting process. We restrict to the case $\theta = 1$. First, we shall consider the mean values of multiplicative functions $g : \mathbf{S}_n \to \mathbf{C}$ defined via $g(\sigma) = \prod_{j=1}^{n} f(j)^{k_j(\sigma)}$, where $f(j)$, $j \geq 1$ are complex numbers that may depend on n or other parameters. Its mean value with respect to the measure $\nu_{n,1}$ equals

$$M_n(g) := \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} g(\sigma) = \sum_{\substack{k_1, \dots, k_n \geq 0 \\ 1k_1 + \dots + nk_n = n}} \prod_{j=1}^{n} \left(\frac{1}{j} f(j)\right)^{k_j} \frac{1}{k_j!}.$$

The behavior of $M_n(g)$ is examined in the next Lemma (see [3]) for large cycles.

Lemma 1. *Let $g : \mathbf{S}_n \to \mathbf{C}$ be a multiplicative function defined via f such that $f(j) = 1$ for all but $j \in J \subset (n/2, n]$. Then*

$$M_n(g) = 1 + \sum_{j \in J} \frac{1}{j}(f(j) - 1).$$

Proof: Observe that, if $k_j \geq 1$ for some $j \in J$, then $1k_1 + \cdots + nk_n = n$ implies $k_j = 1$ and $k_l = 0$ for the remaining $l \neq j$ and $l \in J$. Let $\Sigma_{(0)}$ denotes the sum over all $(k_1, \ldots, k_n)$ satisfying $1k_1 + \cdots + nk_n = n$ and $k_l = 0$ for all $l \in J$, and $\Sigma_{(j)}$ denotes the sum over all $(k_1, \ldots, k_n)$ satisfying $1k_1 + \cdots + nk_n = n$ and $k_j = 1$. Hence

$$\begin{aligned} M_n(g) &= \sum\nolimits_{(0)} \prod_{l=1}^{n} \left(\frac{1}{l}\right)^{k_l} \frac{1}{k_l!} + \sum_{j \in J} f(j) \sum\nolimits_{(j)} \prod_{l=1}^{n} \left(\frac{1}{l}\right)^{k_l} \frac{1}{k_l!} \\ &= 1 + \sum_{j \in J} (f(j) - 1) \sum\nolimits_{(j)} \prod_{l=1}^{n} \left(\frac{1}{l}\right)^{k_l} \frac{1}{k_l!}\,. \end{aligned}$$

The Lemma follows now as the last sum is $(1/j)$.

The characteristic function $\phi_{n,s,t}$ of $H_n(t) - H_n(s)$ for $\frac{1}{2} < s < t \leq 1$ is given by

$$\phi_{n,s,t}(\eta) = M_n\left(e^{i\eta(H_n(t) - H_n(s))}\right), \qquad \eta \in \mathbf{R}.$$

We apply Lemma 1 with $f(j) = \exp(i\eta(j/n)^\rho)$, $\eta \in \mathbf{R}$, to get

$$\begin{aligned} \phi_{n,s,t}(\eta) &= 1 + \sum_{s < (j/n) \leq t} \frac{1}{j}(e^{i\eta(j/n)^\rho} - 1) \\ &\to 1 + \int_s^t \frac{1}{v}\left(e^{i\eta v^\rho} - 1\right) dv \\ &= 1 + \frac{1}{\rho} \int_{s^\rho}^{t^\rho} \frac{1}{u}\left(e^{i\eta u} - 1\right) du =: \phi_{s,t}(\eta). \end{aligned}$$

If the limiting process of H_n has independent increments, then for all $\frac{1}{2} < s < t < 1$, and $\eta \in \mathbf{R}$,

$$\phi_{s,1}(\eta) = \phi_{s,t}(\eta)\,\phi_{t,1}(\eta). \tag{1}$$

Hence for (1) to hold we must have,

$$\left(\int_{s^\rho}^{t^\rho} \frac{1}{u}\left(e^{i\eta u} - 1\right) du\right)\left(\int_{t^\rho}^{1} \frac{1}{u}\left(e^{i\eta u} - 1\right) du\right) = 0$$

for all $\frac{1}{2} < s < t < 1$ and $\eta \in \mathbf{R}$, which is impossible. This shows that the limiting process of H_n is not a process with independent increments.

The weak convergence of processes for general h and non-slowly varying β will be addressed elsewhere. □

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Gutti Jogesh Babu
Department of Statistics,
319 Thomas Building,
The Pennsylvania State University,
University Park, PA 16802, USA

email: babu@stat.psu.edu
http://www.stat.psu.edu/ babu
http://astrostatistics.psu.edu