Stat/Math 414.1 Spring 2008

Midterm 1

February 15, 2008

Your Name:
ID Number:

1) Please turn off your cell phone!
2) Please show all work for full credit. Partial credit will be given if your solution is correct but the answer is wrong.
3) Please do your own work.
4) After handing in the exam sheets, take your homework 3 back and keep your formula sheet.
5) Good luck!

1. (20 points) In a literature class, 30% of students speak French, 35% speak German and 50% of students speak French OR German.

   (a) What is the proportion of students who speak French AND German?

   (b) What is the proportion of students who speak French but NOT German?

   (c) Among the students who speak French, what is the proportion who speak German?

   Solution: Let $A = \{\text{speak French}\}$ and $B = \{\text{speak German}\}$. Then $P(A) = 0.3$, $P(B) = 0.35$, $P(A \cup B) = 0.5$.

   (a) $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.15$;

   (b) $P(A \cap \bar{B}) = P(A) - P(A \cap B) = 0.15$;

   (c) $P(B|A) = P(A \cap B)/P(A) = 0.5$.

2. (20 points) A drawer contains 6 black and 8 white shirts.

(a) How many ways are there to select a sample of 5 shirts from the drawer?

(b) If a sample of 5 shirts is randomly chosen, what is the probability that there are exactly 3 white shirts in the sample?

(c) If a sample of 5 shirts is randomly chosen, what is the probability that there are at least 3 white shirts in the sample?

Solution:

(a) $N(S) = \binom{14}{5} = 2002;$

(b) $P(A) = \binom{6}{2}\binom{8}{3}/\binom{14}{5} = 840/2002 = 0.4196;$

(c) $P(B) = \left[\binom{6}{2}\binom{8}{3} + \binom{6}{1}\binom{8}{4} + \binom{6}{0}\binom{8}{5}\right]/\binom{14}{5} = (840 + 420 + 56)/2002 = 1316/2002 = 0.6574.$

3. (20 points) Seventy percent of the light aircraft that disappear while in flight in a certain country are subsequently discovered. Of the aircraft that are discovered, 60% have an emergency locator, whereas 90% of the aircraft not discovered do not have such a locator. Define events A and B as follows:

   $A = \{\text{light aircraft that disappears is discovered}\}$

   $B = \{\text{light aircraft that disappears has an emergency locator}\}$

   Suppose a light aircraft has disappeared.

   (a) What is the probability that it has an emergency locator and it will not be discovered?

   (b) What is the probability that it has an emergency locator?

   (c) If it has an emergency locator, what is the probability that it will not be discovered?

   Solution: The information we have

$$P(A) = 0.7$$

$$P(B|A) = 0.6$$

$$P(\bar{B}|\bar{A}) = 0.9$$

So,

$$P(B|\bar{A}) = 0.1$$

$$P(\bar{A}) = 0.3$$

Therefore,

(a) $P(B \cap \bar{A}) = P(B|\bar{A}) \cdot P(\bar{A}) = 0.1 \times 0.3 = 0.03$;

(b) $P(B) = P(B|A) \cdot P(A) + P(B|\bar{A}) \cdot P(\bar{A}) = 0.6 \times 0.7 + 0.1 \times 0.3 = 0.45$;

(c) $P(\bar{A}|B) = P(B \cap \bar{A})/P(B) = 0.03/0.45 = 0.067$.

4. (20 points) Thirty percent of credit card holders carry no monthly balance, while 70% do. Of those card holders carrying a balance, 30% have annual income \$20,000 or less, 40% between \$20,001 - \$50,000, and 30% over \$50,000. Of those card holders carrying no balance, 20%, 30%, and 50% have annual incomes in these three respective categories.

(a) What is the probability that a randomly chosen card holder has annual income \$20,000 or less?

(b) If this card holder has an annual income that is \$20,000 or less, what is the probability that (s)he carries a balance?

Solution: Define events as follows:

$A = \{\text{carry a balance}\}$

$B_1 = \{\text{annual income \$20,000 or less}\}$

$B_2 = \{\text{annual income between \$20,001 to \$50,000}\}$

$B_3 = \{\text{annual income over \$50,000}\}$

The information we have

$$
\begin{aligned}
P(A) &= 0.7 \\
P(B_1|A) &= 0.3 \\
P(B_2|A) &= 0.4 \\
P(B_3|A) &= 0.3 \\
P(B_1|\bar{A}) &= 0.2 \\
P(B_2|\bar{A}) &= 0.3 \\
P(B_3|\bar{A}) &= 0.5 \\
P(\bar{B}|\bar{A}) &= 0.9
\end{aligned}
$$

Therefore,

(a) $P(B_1) = P(B_1|A) \cdot P(A) + P(B_1|\bar{A}) \cdot P(\bar{A}) = 0.3 \times 0.7 + 0.2 \times 0.3 = 0.27$;

(b) $P(A|B_1) = P(A \cap B_1)/P(B_1) = P(B_1|A) \cdot P(A)/P(B_1) = 0.3 \times 0.7/0.27 = 0.7778$;

5. (10 points) There are 5 red balls and 7 black balls in a bag. We take these balls out one by one without replacement. Let $A = \{\text{The 6th ball out of the bag is a red ball}\}$. Find the probability of A.

Solution: What we are doing here basically is to put these balls in a row and find the probability that the 6th ball is red. The total number of ways to put these balls in a row is

$$N(S) = \binom{12}{5} or \binom{12}{7}$$

Here I assume all the red balls are the same and all the black balls are the same. If you treat them differently, you need to use permutation and should get the same results.

If the 6th ball the red, the number of possible array is

$$N(A) = \binom{11}{4} or \binom{11}{7}$$

Therefore,

$$P(A) = N(A)/N(S) = 5/12 = 0.4167$$

Note that this probability is the same as the probability that the 1st ball is red.

6. (10 points) Suppose that events A, B and C are mutually independent. And we know that $P(A) = 0.5$, $P(B) = 0.8$, $P(C) = 0.9$. Find the probability that exactly two will occur.

Solution: By the independence condition, we have

$$\begin{aligned} P(\text{exactly two will occur}) &= P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C) \\ &= P(A)P(B)P(\bar{C}) + P(A)P(\bar{B})P(C) + P(\bar{A})P(B)P(C) \\ &= 0.04 + 0.09 + 0.36 \\ &= 0.49 \end{aligned}$$

Bonus: (5 points) Suppose that x_1, x_2, x_3, x_4, x_5 are non-negative integers. How many different solutions are there for the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 15?$$

(For example, (1,0,2,4,8) is a possible solution.)

Solution: Think about it if you have interest. You may first find the number of solutions assume all x's are positive integers.

I will post the solution later.