Homework 13

Due Friday, April 25 in class

Lecture 34 (Friday, April 18)

Do exercises 5.69, 5.70 and 5.75.

For problem 5.69, you may use Theorem 5.9 in the book.

Lecture 35 (Monday, April 21)

Do exercises 5.87, 5.92 and 5.94.

Use Theorem 5.12 in the book.

Lecture 36 (Wednesday, April 23)

Problem 1.

Consider the example given in today's lecture. The probabilities that a certain experiment will last less than 50 hours, between 50 and 90 hours, or more than 90 hours, are $p_1 = 0.2$, $p_2 = 0.5$ and $p_2 = 0.3$, respectively. Suppose we are going to do this experiment 8 times, and set X_1 for the number that last less than 50 hours, X_2 for the number that last between 50 and 90 hours, and X_3 for the number that last more than 50 hours. Then (X_1, X_2, X_3) follows a **multinomial** distribution (Note that $X_1 + X_2 + X_3 = 8$). In class, we already found out $Cov(X_1, X_2)$. Go ahead to find

1. $E(X_1 - 2X_2 + 5X_3)$
2. $Var(X_3)$
3. $Cov(X_2, X_3)$

(Hint: Still let (Y_1, Y_2, Y_3) be the r.v.s for a single experiment ($n = 1$) and use the relationship between X and Y to solve the problem. Also note that r.v.s that come from different experiments are independent of each other because experiments are assumed to be independent with each other.)

Solution: It is easy to start from the joint distribution of (Y_1, Y_2, Y_3) to obtain distributions for Y_1, Y_2, Y_3, Y_2Y_3, etc. And then we have $E(Y_1) = p_1 = 0.2$, $E(Y_2) = p_2 = 0.5$, $E(Y_3) = p_3 = 0.3$, $V(Y_3) = E(Y_3^2) - E^2(Y_3) = p_3 - p_3^2 = 0.21$, $Cov(Y_2, Y_3) = E(Y_2Y_3) - E(Y_2)E(Y_3) = 0 - p_2p_3 = -0.15$.

We know that $X_1 = \sum_{i=1}^{8} Y_{1,i}$, $X_2 = \sum_{i=1}^{8} Y_{2,i}$ $X_3 = \sum_{i=1}^{8} Y_{3,i}$. And $Y_{,i} \perp Y_{,j}$ if $i \neq j$. Therefore,

$E(X_1 - 2X_2 + 5X_3) = E(X_1) - 2E(X_2) + 5E(X_3) = 8(0.2 - 1 + 1.5) = 5.6$

$Var(X_3) = 8Var(Y_3) = 1.68$

$Cov(X_2, X_3) = Cov(\sum_{i=1}^{8} Y_{2,i}, \sum_{j=1}^{8} Y_{3,j}) = \sum_{i=1}^{8} \sum_{j=1}^{8} Cov(Y_{2,i}, Y_{3,j}) = 8Cov(Y_2, Y_3) = -1.2$.

That's all for homework 13.

5.69 Since Y_1 and Y_2 are independent, with $f_1(y_1) = \frac{1}{4}y_1e^{-y_1/2}$ and $f_2(y_2) = \frac{1}{2}e^{-y_2/2}$,

$$E\left(\frac{Y_2}{Y_1}\right) = E\left(\frac{1}{Y_1}\right)E(Y_2) = \frac{1}{8}\int_0^\infty e^{-y_1/2}\,dy_1 \int_0^\infty y_2e^{-y_2/2}\,dy_2$$
$$= \frac{1}{8}\left[-2e^{-y_1/2}\right]_0^\infty(4) = \frac{1}{4}(4) = 1$$

since the second integral is the variable factor of a gamma distribution with $\alpha = 2$, $\beta = 2$ and integrates to $\Gamma(2)2^2 = 4$.

5.70 The marginal distribution of Y_1 is $f_1(y_1) = 1$ for $0 \le y_1 \le 1$, so that $E(Y_1) = \int_0^1 y_1\,dy_1 = \frac{1}{2}$. Using the joint distribution of Y_1 and Y_2, we obtain

$$E(Y_2) = \int_0^1\int_0^{y_1} \frac{y_2}{y_1}\,dy_2\,dy_1 = \int_0^1 \frac{y_1^2}{2y_1}\,dy_1 = \frac{y_1^2}{4}\Big]_0^1 = \frac{1}{4}$$

Thus, $E(Y_1 - Y_2) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

5.75 $\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2)$.

$$E(Y_1Y_2) = \sum_{y_1}\sum_{y_2} y_1y_2p(y_1, y_2) = (0)(0)\left(\tfrac{1}{9}\right) + (1)(0)\left(\tfrac{2}{9}\right) + (2)(0)\left(\tfrac{1}{9}\right) + (0)(1)\left(\tfrac{2}{9}\right)$$
$$+ (1)(1)\left(\tfrac{2}{9}\right) + (0)(2)\left(\tfrac{1}{9}\right) = \tfrac{2}{9}.$$

Since Y_1 and Y_2 are both binomial with $n = 2$ and $p = \frac{1}{3}$,

$$E(Y_1) = E(Y_2) = 2\left(\tfrac{1}{3}\right) = \tfrac{2}{3}.$$

Thus, $\text{Cov}(Y_1, Y_2) = \left(\frac{2}{9}\right) - \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = -\frac{2}{9}$.

No, as value of Y_1 increases, value of Y_2 tends to decrease.

5.87 Refer to Theorem 5.12.

$$E(3Y_1 + 4Y_2 - 6Y_3) = 3(2) + 4(-1) - 6(4) = -22$$
$$V(3Y_1 + 4Y_2 - 6Y_3) = 9(4) + 16(6) + 36(8) + (2)(3)(4)(1) + (2)(3)(-6)(1) + 2(4)(-6)(0) = 480$$

5.92 From Exercise 5.29, $f_1(y_1) = y_1e^{-y_1}$, which is a gamma distribution with $\alpha = 2, \beta = 1$. Hence $E(Y_1) = 2(1) = 2$ and $V(Y_1) = \alpha\beta^2 = 2$.

$$f_2(y_2) = \int_{y_2}^\infty e^{-y_1}\,dy_1 = -e^{-y_1}\Big]_{y_2}^\infty = e^{-y_2}$$

which has a gamma distribution with $\alpha = \beta = 1$. Hence $E(Y_2) = V(Y_2) = 1$. Finally,

$$E(Y_1Y_2) = \int_0^\infty\int_0^{y_1} y_1y_2e^{-y_1}\,dy_2\,dy_1 = \int_0^\infty \frac{y_1^3}{2}e^{-y_1}\,dy_1 = \frac{\Gamma(4)1^4}{2} = 3$$

$$\text{Cov}(Y_1, Y_2) = 3 - (1)(2) = 1 \qquad E(Y_1 - Y_2) = 2 - 1 = 1$$
$$V(Y_1 - Y_2) = 2 + 1 - 2(1) = 1$$

It is unlikely that a customer would spend more than 4 minutes at the service window because this is 3 standard deviations above the mean.

5.94 Notice $f(y_1, y_2) = \frac{1}{4}y_1e^{-y_1/2}\frac{1}{2}e^{-y_2/2}$ for $0 \le y_1, y_2 \le \infty$. Therefore Y_1 and Y_2 are independent. Notice also this implies that Y_1 is distributed as a gamma with $\alpha = 2$ and $\beta = 2$, and Y_2 is distributed as an exponential with $\beta = 2$. Hence $E(Y_2) = 2$ and $V(Y_2) = 4$, and $E(Y_1) = 4$, $V(Y_1) = 8$. Since Y_1 and Y_2 are independent $\text{Cov}(Y_1, Y_2) = 0$. Then

$$E(C) = 50 + 2(4) + 4(2) = 66 \qquad V(C) = 4(8) + 16(4) + 0 = 96$$