6.6 **a.** Consider $U = Y_1 - Y_2$, where Y_1 and Y_2 lie in the large triangle shown in Figure 6.1. $F_U(u)$ can be found directly by integrating the joint density over various regions, which will change depending on the value taken by u. Note that the line $Y_1 - Y_2 = u$ or $Y_1 = Y_2 + u$ is a set of parallel lines having intercept u on the Y_1 axis.

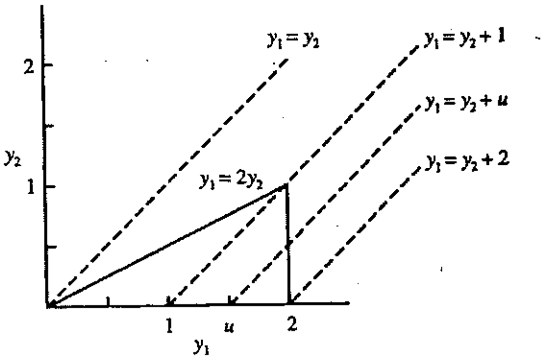

Figure 6.1

(1) For $u \le 0$,
$F_U(u) = P(Y_1 - Y_2 \le u) = 0.$

(2) For $0 \le u \le 1$,

$$F_U(u) = \int_0^u \int_{2y_2}^{y_2+u} dy_1\, dy_2 = \int_0^u (u - y_2)\, dy_2 = \tfrac{u^2}{2}$$

(3) For $1 \le u \le 2$,

$$F_U(u) = 1 - \int_0^{2-u} \int_{y_2+u}^{2} dy_1\, dy_2 = 1 - \int_0^{2-u} (2 - y_2 - u)\, dy_2 = 1 - \tfrac{(2-u)^2}{2}$$

(4) For $u > 2$, $F_U(u) = 1$.
Differentiating with respect to u, we have

$$f_U(u) = \begin{cases} u, & 0 \le u \le 1 \\ 2-u, & 1 \le u \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

b. $E(U) = \int_0^1 u^2\, du + \int_1^2 (2u - u^2)\, du = \frac{1}{3} + \left[u^2 - \frac{u^3}{3}\right]_1^2 = 1$

6.19 **a.** If $U = 2Y - 1$, then $Y = \frac{U+1}{2}$. Differentiating, we have $\frac{dy}{du} = \frac{1}{2}$ and

$$f_U(u) = \tfrac{1}{2}(2)\left(1 - \tfrac{u+1}{2}\right) = \tfrac{1-u}{2}$$

for $0 \le \frac{u+1}{2} \le 1$ or $-1 \le u \le 1$.

b. If $U = 1 - 2Y$, then $Y = \frac{1-U}{2}$, and $\frac{dy}{du} = \frac{1}{2}$. Then

$$f_U(u) = \tfrac{1}{2}(2)\left(1 - \tfrac{1-u}{2}\right) = \tfrac{1+u}{2}$$

for $0 \le \frac{1-u}{2} \le 1$ or $-1 \le u \le 1$.

c. If $U = Y^2$, then $Y = \sqrt{U}$ and $\frac{dy}{du} = \frac{1}{2\sqrt{U}}$. Then (since $u = y^2$ is increasing for $y > 0$)

$$f_U(u) = \tfrac{1}{2\sqrt{u}}(2)\left(1 - \sqrt{u}\right) = \tfrac{1-\sqrt{u}}{\sqrt{u}}$$

for $0 \le \sqrt{u} \le 1$ or $0 \le u \le 1$.

6.24 $f(y) = 1 \quad$ for $0 \le y \le 1$

$U = -2 \ln Y$; solving for y,

$\ln y = -\frac{u}{2}$ which gives $y = e^{-u/2}$.

$\frac{dy}{du} = \left(-\frac{1}{2}\right) e^{-u/2}$.

Thus,

$$f_u(u) = f(y)\left|\tfrac{dy}{du}\right| = 1\left|\left(-\tfrac{1}{2}\right) e^{-u/2}\right| = \left(\tfrac{1}{2}\right) e^{-u/2} \qquad \text{for } u > 0$$

which is exponential with $\beta = 2$.

6.27 Similar to Exercise 6.21. Fix $Y_1 = y_1$. Then if $U = \frac{Y_2}{y_1}$, $Y_2 = y_1 U$ and $\left|\frac{dy_2}{du}\right| = y_1$

The joint density of Y_1 and U is

$$f(y_1, u) = \tfrac{1}{8} y_1^2 e^{-(y_1 + uy_1)/2} = \tfrac{1}{8} y_1^2 e^{-y_1(1+u)/2} \qquad y_1 > 0,\ u > 0$$

Integrating over all possible values of y_1, we have

$$f_U(u) = \tfrac{1}{8}\int_0^\infty y_1^2 e^{-y_1(1+u)/2}\, dy_1 = \frac{\Gamma(3)\left[\frac{2}{1+u}\right]^3}{8} = \frac{2}{(1+u)^3} \qquad u > 0$$

since the variable part of the integrand is that of a gamma variable with $\alpha = 3$, $\beta = \frac{2}{1+u}$.

6.30 **a.** If $U = Y^2$, then $Y = \sqrt{U}$ and $\left|\frac{dy}{du}\right| = \frac{1}{2\sqrt{u}}$. Then

$$f_U(u) = \tfrac{2\sqrt{u}}{\theta} \times e^{-u/\theta} \times \tfrac{1}{2\sqrt{u}} = \tfrac{1}{\theta} e^{-u/\theta} \qquad u \ge 0$$

which is an exponential density with mean θ.

b. From part **a.**, $E(Y^2) = E(U) = \theta$. Further, integrating directly, we have

$$E(Y) = E\left(U^{1/2}\right) = \int_0^\infty \frac{u^{1/2} e^{-u/\theta}}{\theta}\, du = \frac{\Gamma\left(\frac{3}{2}\right)\theta^{3/2}}{\theta} = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right)\theta^{1/2} = \frac{\sqrt{\pi\theta}}{2}$$

Then $V(Y) = E(Y^2) - [E(Y)]^2 = \theta - \left(\frac{\pi\theta}{4}\right) = \theta\left[1 - \left(\frac{\pi}{4}\right)\right]$.

6.32 By independence we know that

$$f(y_1, y_2) = \frac{4y_1y_2}{\theta^2}e^{-(y_1^2+y_2^2)/\theta} \text{ when } y_1 > 0 \text{ and } y_2 > 0.$$

Let $U = Y_1^2 + Y_2^2$. Now fix a value of Y_1 at y_1, say. Then set $U = y_1^2 + Y_2^2$ so that $y_2 = \sqrt{u - y_1^2}$. Note that

$$\frac{dh^{-1}(u)}{du} = \frac{1}{2\sqrt{u-y_1^2}}.$$

Then we have

$$g(y_1, u) = \frac{4y_1\sqrt{u-y_1^2}}{\theta^2}e^{-u/\theta}\frac{1}{2\sqrt{u-y_1^2}} = \frac{2e^{-u/\theta}}{\theta^2}y_1$$

for $0 < y_1 < \sqrt{u}$. Then

$$f_U(u) = \int_0^{\sqrt{u}} \frac{2e^{-u/\theta}}{\theta^2}y_1 dy_1 = \frac{1}{\theta^2}ue^{-u/\theta}.$$

for $u > 0$. That is, $U \sim \text{Gamma}(2, \theta)$.

6.44 **a.** Because Y_1 and Y_2 are Poisson random variables,

$$m_{Y_1}(t) = e^{\lambda_1(e^t-1)}$$

and

$$m_{Y_2}(t) = e^{\lambda_2(e^t-1)}.$$

So that $m_{Y_1+Y_2}(t) = \exp[-(\lambda_1 + \lambda_2)(1 - e^t)]$; which is the moment-generating function of a Poisson random variable with mean $\lambda_1 + \lambda_2$. (Recall that exp( ) is simply a convenient way to write $e^{(\ )}$). By Theorem 6.1, then,

$$P(Y_1 + Y_2 = k) = \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^k}{k!}$$

for $k = 0, 1, 2, \ldots$

b. By definition,

$$P(Y_1 = k|Y_1 + Y_2 = m) = \frac{P(Y_1=k, Y_1+Y_2=m)}{P(Y_1+Y_2=m)} = \frac{P(Y_1=k, Y_2=m-k)}{P(Y_1+Y_2=m)}$$

$$= \frac{\left[\frac{e^{-\lambda_1}\lambda_1^k}{k!}\right]\left[\frac{e^{-\lambda_2}\lambda_2^{m-k}}{(m-k)!}\right]}{\left[e^{-(\lambda_1+\lambda_2)}\left(\frac{(\lambda_1+\lambda_2)^m}{m!}\right)\right]} = \binom{m}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{m-k} \quad k = 0, 1, 2, \ldots, m$$

which is the probability distribution function for a binomial random variable with parameters m and $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

6.50 **a.** Recall that $m_{W_i}(t) = \frac{pe^t}{(1-qe^t)}$. Using Theorem 6.2, we have

$$m_Y(t) = \prod_{i=1}^{r} m_{W_i}(t) = \left(\frac{pe^t}{1-qe^t}\right)^r$$

b. Differentiating with respect to t, we have

$$E(Y) = m'(0) = r\left(\frac{pe^t}{1-qe^t}\right)^{r-1} \times \left.\frac{pe^t}{(1-qe^t)^2}\right|_{t=0} = \frac{r}{p}.$$

Further,

$$E(Y^2) = m''(0) = \frac{d}{dt}\left.\frac{r(pe^t)^r}{(1-qe^t)^{r+1}}\right|_{t=0}$$

$$= \left.\frac{(1-qe^t)^{r+1}r^2pe^t(pe^t)^{r-1} - r(pe^t)^r(r+1)(-qe^t)(1-qe^t)^r}{(1-qe^t)^{2(r+1)}}\right|_{t=0}$$

$$= \frac{pr^2+r(r+1)q}{p^2}$$

Then

$$V(Y) = \frac{pr^2+r(r+1)q-r^2}{p^2} = \frac{rq}{p^2}.$$

c. This is similar to exercises 6.45 and 6.46.

$$P(W_1 = k|\sum W_i = m) = \frac{P(W_1 = k, \sum W_i = m)}{P(\sum W_i = m)}$$

$$= \frac{P(W_1 = k, \sum_2^r W_i = m - k)}{P(\sum W_i = m)}$$

$$= \frac{P(W_1 = k)P(\sum_2^r W_i = m - k)}{P(\sum W_i = m)}$$

$$= \frac{\binom{m-k-1}{r-2}}{\binom{m-1}{}}.$$

P_{350}. 7.30 By central limit Theorem,

$$P(|\bar{Y}-\mu|\le 1) = P(-1\le \bar{Y}-\mu\le 1)$$

$$= P\left(-1\cdot\frac{\sqrt{n}}{\sigma} \le \frac{\sqrt{n}(\bar{Y}-\mu)}{\sigma} \le \frac{\sqrt{n}\cdot 1}{\sigma}\right)$$

$$\approx P\left(-\frac{\sqrt{n}}{\sigma} \le Z \le \frac{\sqrt{n}}{\sigma}\right)$$ Z is standard Normal

$$= P\left(-\frac{\sqrt{100}}{10} \le Z \le \frac{\sqrt{100}}{10}\right) = P(-1\le Z\le 1) \approx 68\%$$

7.32

(a). $$P(199\le \bar{Y}\le 202) = P\left(\frac{\sqrt{n}(199-\mu)}{\sigma} \le \sqrt{n}\frac{\bar{Y}-\mu}{\sigma} \le \frac{\sqrt{n}(202-\mu)}{\sigma}\right)$$

$$\approx P\left(\frac{\sqrt{n}(199-\mu)}{\sigma} \le Z \le \frac{\sqrt{n}(202-\mu)}{\sigma}\right)$$ Z: standard normal

$$= P\left(\frac{\sqrt{25}(199-200)}{10} \le Z \le \frac{\sqrt{25}(202-200)}{10}\right)$$

$$= P(-0.5 \le Z \le 1) \approx 0.533$$

(b). $$P(Y_1+\cdots+Y_n \le 5100) = P\left(\bar{Y} \le \frac{5100}{n}\right) = P\left(\sqrt{n}\frac{(\bar{Y}-\mu)}{\sigma} \le \frac{\sqrt{n}}{\sigma}\left(\frac{5100}{n}-\mu\right)\right)$$

$$\approx P\left(Z \le \frac{\sqrt{n}}{\sigma}\left(\frac{5100}{n}-\mu\right)\right) = P\left(Z\le \frac{\sqrt{25}}{10}\left(\frac{5100}{25}-200\right)\right)$$

$$= P(Z\le 2) \approx 97.7\%$$