Quiz 1 with Solution

Problem 1. 6 people are to be seated in a row of 6 seats.

1. What is the probability that Mary and Jim will sit next to each other?
2. What is the probability that Mary and Jim will sit next to each other, and Diana and Steven will sit together?

Problem 2. In the game Texas HoldEm, it is well known that Flush is a better hand than Straight.

A flush is a poker hand such as $Q\clubsuit$ $7\clubsuit$ $J\clubsuit$ $4\clubsuit$ $6\clubsuit$, which contains five cards of the same suit, not in rank sequence.

A straight is a poker hand such as $5\clubsuit$ $8\diamondsuit$ $6\spadesuit$ $9\heartsuit$ $7\spadesuit$, which contains five cards of sequential rank but in more than one suit.

If I randomly deal 5 cards to you from a full deck of 52, what is the probability that

1. you will get a hand of flush?
2. you will get a hand of straight?
3. you will get a hand with no pair in it?

Solution for P1:

1. The total number of ways of ordering 6 people is $N = P_6^6$. Denote by A the event that Mary and Jim will sit next to each other. To compute the number of sample points in A, we can do it in two steps:

   Step 1. Because Mary and Jim will sit together, we can bond them together to form a superman. Now we have 4 remaining people and the superman. The number of ways of ordering these 5 people (4 original people and one superman) is $N_1 = P_5^5$.

   Step 2. Within the superman, we can have two orderings: (Mary, Jim) and (Jim, Mary).

   Therefore, the number of sample points in A is $N_A = N_1 \times 2 = P_5^5 \times 2$. The probability of A is
$$\frac{N_A}{N} = \frac{P_5^5 \times 2}{P_6^6} = \frac{5! \times 2}{6!} = \frac{1}{3}.$$

2. Again, the total number of ways of ordering 6 people is $N = P_6^6$. Denote by B the event that Mary and Jim will sit next to each other, and Diana and Steven will sit together. To compute the number of sample points in B, we can do it in three steps:

   Step 1. Because Mary and Jim, Diana and Steven will sit together, we can bond Mary and Jim together to form a superman S_1, and bond Diana and Steven together to form another superman S_2. Now we have 2 remaining people and the two supermen. The number of ways of ordering these 4 people (2 original people and two supermen) is $N_2 = P_4^4$.

   Step 2. Within the superman S_1, we can have two orderings: (Mary, Jim) and (Jim, Mary).

   Step 3. Within the superman S_2, we can have two orderings: (Diana, Steven) and (Steven, Diana).

   Therefore, the number of sample points in B is $N_B = N_2 \times 2 \times 2 = P_4^4 \times 2 \times 2$. The probability of B is
$$\frac{N_B}{N} = \frac{P_4^4 \times 2 \times 2}{P_6^6} = \frac{4! \times 2 \times 2}{6!} = \frac{2}{15}.$$

Solution for P2:

The total number of ways to choose 5 cards from a deck of 52 is $\binom{52}{5} = 2598960$.

1. Number of hands that five cards in the same suit: $\binom{13}{5}\binom{4}{1}$;

   Number of hands that five cards in the same suit AND in rank sequence: $\binom{10}{1}\binom{4}{1}$;

   The number of hands of flush is: $\binom{13}{5}\binom{4}{1} - \binom{10}{1}\binom{4}{1} = 5108$.

   So the probability that you will get a hand of flush is

$$\frac{\binom{13}{5}\binom{4}{1} - \binom{10}{1}\binom{4}{1}}{\binom{52}{5}} = \frac{5108}{2598960} = 0.00197.$$

2. Number of hands that five cards in rank sequence: $4^5 \cdot \binom{10}{1}$;

   Number of hands that five cards in rank sequence AND in the same suit: $\binom{10}{1}\binom{4}{1}$;

   The number of hands of straight is: $4^5 \cdot \binom{10}{1} - \binom{10}{1}\binom{4}{1} = 10200$.

   So the probability that you will get a hand of straight is

$$\frac{4^5 \cdot \binom{10}{1} - \binom{10}{1}\binom{4}{1}}{\binom{52}{5}} = \frac{10200}{2598960} = 0.00392.$$

   Therefore, you can see the reason why flush is a better hand than straight.

3. Consider rank first:

   Number of hands that five cards without same rank and not in rank sequence: $\binom{13}{5} - 10$;

   Then consider suit :

   Number of hands that five cards not in the same suit: $\binom{4}{1}^5 - 4$;

   The number of hands of no pair is: $\left[\binom{13}{5} - 10\right]\left[\binom{4}{1}^5 - 4\right] = 1302540.$

So the probability that you will get a hand with no pair is

$$\frac{\left[\binom{13}{5} - 10\right]\left[\binom{4}{1}^5 - 4\right]}{\binom{52}{5}} = \frac{1302540}{2598960} = 0.5012.$$