

This closed-book final is worth 20 points. You have 110 minutes. Note that not all questions are weighted equally; allot your time accordingly. You are allowed to use two sheets (double-sided) of your own notes. Write your answers on separate pages, and be sure that your name appears on each page you turn in.

Problem 1. Suppose that X_1, \dots, X_n is an iid sequence with X_i geometrically distributed:

$$P(X_i = k) = p(1 - p)^k, \quad k = 0, 1, \dots$$

for some $p \in (0, 1)$. Note that $n\bar{X}_n$ has a negative binomial distribution.

The geometric distribution above has mean $(1 - p)/p$ and variance $(1 - p)/p^2$.

(a) If we suppose that the prior distribution on p is $\text{Beta}(\frac{1}{2}, \frac{1}{2})$, find (with proof) a and b such that the Bayes estimator of p is

$$\delta_n = \frac{n + a}{n + n\bar{X}_n + b}$$

The Beta (a, b) distribution has density $\Gamma(a + b)x^{a-1}(1 - x)^{b-1}I\{0 < x < 1\}/\{\Gamma(a)\Gamma(b)\}$, expectation $a/(a + b)$, and variance $ab/\{(a + b)^2(a + b + 1)\}$.

(b) Verify directly that the Bayes estimator in part (a) is an efficient estimator of p . (If you can't answer part (a), solve this part using $a = b = 2$.)

Problem 2. Suppose that for some $\theta \in R$, X has the density

$$f_\theta(x) = \frac{1}{\Gamma(\frac{1}{4})} \exp\left\{-\frac{1}{16}(x - \theta)^4\right\}$$

for all real x . Then X has mean θ and variance $4\Gamma(\frac{3}{4})/\Gamma(\frac{1}{4}) \approx 1.3520$.

(a) Find the Fisher information $I(\theta)$. Furthermore, by equating two different expressions for $I(\theta)$, evaluate $E(X - \theta)^6$.

(b) Let R_n be the Rao score statistic $\ell'_n(0)/\sqrt{nI(0)}$ for testing the hypothesis $H_0 : \theta = 0$. Prove that $R_n \xrightarrow{\mathcal{L}} N(0, 1)$ under H_0 .

Problem 3. Let X_1, \dots, X_n be an independent sample from an exponential distribution with mean λ , and Y_1, \dots, Y_n be an independent sample from an exponential distribution with mean μ . Assume that X_i and Y_i are independent. We are interested in testing the hypothesis $H_0 : \lambda = \mu$ versus $H_1 : \lambda > \mu$. Consider the statistic

$$T_n = 2 \sum_{i=1}^n (I_i - 1/2) / \sqrt{n},$$

where I_i is the indicator variable $I_i = I(X_i > Y_i)$.

The exponential distribution with mean θ has density $f_\theta(x) = \frac{1}{\theta} \exp(-x/\theta)$ and cdf $F_\theta(x) = 1 - \exp(-x/\theta)$ for $x > 0$.

(a) Derive the asymptotic distribution of T_n under the null hypothesis.

(b) Use the Lindeberg-Feller Theorem to show that, under the alternative hypothesis $(\lambda_n, \mu_n) = (\mu + n^{-1/2}\delta, \mu)$, where $\delta > 0$ is a constant,

$$\frac{\sum_{i=1}^n (I_i - \rho_n)}{\sqrt{n\rho_n(1 - \rho_n)}} \xrightarrow{\mathcal{L}} N(0, 1), \quad \text{where } \rho_n = \frac{\lambda_n}{\lambda_n + \mu_n} = \frac{\mu + n^{-1/2}\delta}{2\mu + n^{-1/2}\delta}.$$

(c) Using the conclusion of part (b), derive the asymptotic distribution of T_n under the alternative hypotheses specified in (b).

Problem 4. Let X_1, \dots, X_n be an iid random sample from an exponential distribution with mean θ . Define

$$S_n^2 = \frac{1}{n} \sum_{i=1}^n X_i^2 - (\bar{X}_n)^2.$$

For the sake of convenience, define $Y_i = X_i^2$. Then $S_n^2 = \bar{Y}_n - (\bar{X}_n)^2$.

The exponential distribution with mean θ has k th moment $E X^k = k!\theta^k$.

(a) Derive the joint asymptotic distribution of \bar{X}_n and \bar{Y}_n .

(b) Show that

$$\sqrt{n} \left\{ \begin{pmatrix} \bar{X}_n \\ S_n^2 \end{pmatrix} - \begin{pmatrix} \theta \\ \theta^2 \end{pmatrix} \right\} \xrightarrow{\mathcal{L}} N \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \theta^2 & 2\theta^3 \\ 2\theta^3 & 8\theta^4 \end{pmatrix} \right\}.$$

(c) Define the coefficient of variation to be

$$C_n = \frac{\sqrt{S_n^2}}{\bar{X}_n}.$$

Show that $\sqrt{n}(C_n - 1) \xrightarrow{\mathcal{L}} N(0, 1)$.