Stat 597A: Asymptotic Tools — Fall 2002
Final Exam Review Questions — December, 2002

Problem 1. If c is a constant such that $X_n \xrightarrow{\mathcal{L}} c$, prove that $X_n \xrightarrow{P} c$.

Problem 2. Suppose $X_1, X_2, \ldots$ are iid Bernoulli(p) random variables. Let $S_n = \sum_{i=2}^n X_i X_{i-1}$. Find (with justification) the asymptotic distribution of S_n/n.

Problem 3. Either construct an example of each of the following or prove that no such example exists:

(a) [1 point] A sequence $X_1, X_2, \ldots$ of random variables, a random variable X, and a constant x_0 such that $X_n \xrightarrow{\mathcal{L}} X$ and $F_n(x_0) \not\to F(x_0)$.

(b) [1 point] A sequence $X_1, X_2, \ldots$ of random variables such that $X_n \xrightarrow{P} 0$ and $\mathrm{E}(X_n) \to 1$.

Problem 4. Let $X_1, \ldots, X_n$ be iid from a continuous symmetric distribution centered at 0. Suppose $(Y_1, \ldots, Y_n)$ is a permutation of $(X_1, \ldots, X_n)$ satisfying $|Y_1| < |Y_2| < \cdots < |Y_n|$; that is, the Y_i are the X_i arranged in order of increasing absolute value.

Let $W_n = \sum_{i=1}^n iI\{Y_i > 0\}$ be the usual signed-rank statistic. Derive the asymptotic distribution of W_n, justifying your steps.

[We have seen at least two ways to do this. You may of course use any valid method you choose.]

Problem 5. Suppose $X_1, \ldots, X_n$ are iid random variables with cdf $F(x)$. Let $\tilde{X}_n$ denote the sample median. Suppose we wish to estimate $h(F) = \mathrm{Var}(\tilde{X}_n) < \infty$. We use a bootstrap scheme in which we draw B random samples of size n from $\hat{F}_n$, the empirical cdf, and let M_i be the sample median of the ith sample, $i = 1, \ldots, B$.

If $\overline{M}_B$ denotes $(1/B)\sum_{i=1}^B M_i$, explain (with justification) what happens to

$$\frac{1}{B}\sum_{i=1}^{B}(M_i - \overline{M}_B)^2$$

as $B \to \infty$.

Problem 6. Suppose $X_1, X_2, \ldots$ are iid Poisson(θ) random variables. Find the asymptotic distribution of $(S_n - \mathrm{E}\, S_n)/\sqrt{\mathrm{Var}\, S_n}$, where

$$S_n = \sum_{i=2}^{n+1} X_i I\{X_{i-1} = 0\}.$$

The Poisson(θ) distribution has expectation θ, variance θ, and mass function $p(x) = e^{-\theta}\theta^x/x!$ for x a nonnegative integer.

Problem 7. Suppose that $X_1, \ldots, X_n$ are iid with

$$P(X_i = 0) = \theta \qquad \text{and} \qquad P\left(X_i = -\sqrt{1-\theta}\right) = P\left(X_i = \sqrt{1-\theta}\right) = \frac{1-\theta}{2}.$$

Define $Y_i = I\{X_i = 0\}$.

Let $\underline{Z}^{(i)} = (X_i, Y_i)$. Find a function $\underline{g}(x, y) = [g_1(x, y), g_2(x, y)]$ that is a variance-stabilizing transformation in the sense that the asymptotic distribution of $g\left[\overline{\underline{Z}}\right] - g\left[\mathrm{E}(\underline{Z}^{(i)})\right]$ has a covariance matrix that doesn't depend on θ, where $\overline{\underline{Z}}$ is the sample mean of the $\underline{Z}^{(i)}$.

You may find it helpful to know that

$$\frac{d}{dt} 2 \sin^{-1}(\sqrt{t}) \quad = \quad \frac{1}{\sqrt{t(1-t)}}.$$

Problem 8. Let $X_1, \ldots, X_n$ be an iid sample from Beta$(\alpha, 1)$; that is, $f_\alpha(x) = \alpha x^{\alpha-1}$. You may assume without proof that all relevant regularity conditions apply to the beta distribution.

The Beta(α, β) *distribution has expectation* $\alpha/(\alpha+\beta)$, *variance* $\alpha\beta/[(1+\alpha+\beta)(\alpha+\beta)^2]$, *and density* $\Gamma(\alpha+\beta)x^{\alpha-1}(1-x)^{\beta-1}I\{0 < x < 1\}/[\Gamma(\alpha)\Gamma(\beta)]$.

(a) Compute a Wald test statistic W_n that is asymptotically standard normal under $H_0 : \alpha = \alpha_0$.

(b) Compute a Rao score test statistic R_n that is asymptotically standard normal under $H_0 : \alpha = \alpha_0$.

Problem 9. Suppose $X_1, X_2, \ldots$ are independent with $X_i \sim$ Beta(α_i, α_i), where $0 < \alpha_i < 2$. Prove that

$$\frac{\sum_{i=1}^n \left(X_i - \frac{1}{2}\right)}{\sqrt{s_n^2}} \xrightarrow{\mathcal{L}} N(0, 1),$$

where $s_n^2 = \sum_{i=1}^n \mathrm{Var}(X_i)$, by verifying the Lindeberg condition or the Lyapunov condition.

The Beta(α, β) *distribution has expectation* $\alpha/(\alpha+\beta)$, *variance* $\alpha\beta/[(1+\alpha+\beta)(\alpha+\beta)^2]$, *and density* $\Gamma(\alpha+\beta)x^{\alpha-1}(1-x)^{\beta-1}I\{0 < x < 1\}/[\Gamma(\alpha)\Gamma(\beta)]$.

Problem 10. Let $X_1, \ldots, X_n$ be an iid sample from Poisson(θ). Throughout this problem, you may assume that the Poisson distribution satisfies all relevant regularity conditions.

(a) Show that the Jeffreys prior on $(0, \infty)$ is the improper prior density $\lambda(\theta) = 1/\sqrt{\theta}$. To do this, it suffices to show that $\lambda(\theta)$ is proportional to $\sqrt{I(\theta)}$.

The Poisson(θ) *distribution has expectation* θ, *variance* θ, *and mass function* $p(x) = e^{-\theta}\theta^x/x!$ *for* x *a nonnegative integer.*

(b) Show that with the improper Jeffreys prior $\lambda(\theta) = 1/\sqrt{\theta}$, the posterior distribution of θ is gamma. Find the Bayes estimator $\delta_n = \mathrm{E}(\theta \mid X_1, \ldots, X_n)$ using this prior. Finally, give the asymptotic distribution of $\sqrt{n}(\delta_n - \theta)$.

The Gamma(α, β) *distribution has expectation* α/β, *variance* α/β^2, *and density* $\beta^\alpha x^{\alpha-1}e^{-\beta x}I\{x > 0\}/\Gamma(\alpha)$.