

*Constant Proportion Debt Obligations, Zeno's Paradox,  
and the Spectacular Financial Crisis of 2008*

Donald Richards  
Penn State University

Joint work with Hein Hundal (Momentum Investments Services)

Sunday, May 25, 2008

---

Wall Street Journal articles on Moody's Investors Service

High ratings on constant proportion debt obligations (CPDOs)

Concerns raised by the SEC and the Congress about CPDOs

“New debt products test Moody's methods,” WSJ, May 22, 2008

“In a 2006 ‘primer’ on CPDOs ... Moody's used an analogy that compared the product to a ‘coin-toss game.’ The strategy ‘is based on the notion that if ‘heads’ has appeared more frequently than expected, it is less likely to continue appearing,’ the report said.”

## The Origin of CPDOs

---

8/06: CPDOs designed by quants at ABN Amro Securities

A new financial product in the synthetic CDO market

11/06: ABN Amro's "SURF" CPDO deal closed, € 1.35 B

2/07: SURF deal awarded:

Risk Magazine's "Deal of the Year" award for 2006

Int'l. Financial Law Review Award of 2006:

"Innovation of the Year ... ultimate recognition for achievement in global capital markets"

Euromoney: "one of six 'Deals of the Year 2006' "

Bear, Stearns: “The Holy Grail of structured finance”

3/07: Estimated € 1.5-5.2B of CPDOs outstanding

11/07: Tacit approval of CPDOs by the **IMF, ECB, BoE**

11/07: CPDO defaults begin; rated Aaa by Moody’s in 3/07

2/08: Moody’s had rated about € 2.6B (notional) of CPDOs

2/08: Moody’s downgraded several CPDOs, including SURFs

4/08: The **BIS** critiqued CPDOs

## CPDOs had problems from the start

---

2007: Fitch Ratings and Dominion Bond Rating Service criticized CPDOs

Moody's admitted to mistakes in their ratings, attributed the problems to software errors, and downgraded many CPDOs

2008: More CPDOs default; Irish Stock Exchange delistings

Why did CPDOs crash so much faster than usual?

Misconceptions of basic probability theory by CPDO designers

“At bottom, my critique is pretty simple-minded: Nobody pays much attention to the assumptions, and the technology tends to overwhelm common sense.”

– David Freedman (1987)

## What is a CPDO?

---

A special purpose vehicle (SPV) sells notes (bonds)

The SPV invests the proceeds in risk-free securities (U.S. Treasury bonds)

SPV concurrently makes a leveraged sale of an index of credit-default swaps (e.g., iTraxx, CDX)

Some SPV's used an index of high-grade corporate bonds

After 6 months, the SPV settles the old trade and then repeats the process

The level of leveraging depends on the difference between the the SPV's future liabilities and its net asset value

CPDOs designed to pay a coupon rate 2% **above** U.S. T-bonds

Hmmm .... this sounds like a free meal, and yet ...

S&P and Moody's rated CPDOs at AAA (after discussions with ABN Amro)

A free meal on Wall Street contradicts the Efficient Market Hypothesis

It would be strange if CPDO inventors, as believers in EMH, could devise a free meal that contradicts EMH

## Moody's CPDO Primer: A coin-toss model

CPDO starts with a stake of \$1 =  $\underbrace{0.1}_{\text{capital } (\gamma)} + \underbrace{0.9}_{\text{bank loan } (1-\gamma)}$

Goal: To double the stake of \$1 (Cash-In Event)

A return of 1,000% on CPDO's capital after repaying bank

On each toss, CPDO bets 1% of: \$2 minus the current stake

Bank makes a margin call if the stake falls to or below 90 cents  
(Cash-Out Event)

“Such a strategy is based on the notion that if ‘heads’ has appeared more frequently than expected, it is less likely to continue appearing, and similarly for ‘tails.’ ”

## Problems with the coin-toss model

---

“Mean-reversion” is “gambler’s fallacy,” a misbelief that long-run relative frequency of a random event must occur in the short run

Many casino gamblers believe that a run of good luck will soon follow a run of bad luck

Tversky & Kahneman (1971):

“... misconception of the ... laws of chance. The gambler feels that the fairness of the coin entitles him to expect that deviation in one direction will soon be cancelled by a corresponding deviation in the other.”

Classical theorem: If you toss a fair coin indefinitely then any given string of heads or tails will occur infinitely often

## The coin-toss strategy ...

---

Requires a larger bet after a loss and a smaller bet after a win

Opposite to Kelly's criterion, Whitworth's formula, optimal strategies (Dubins & Savage; Maitra & Sudderth)

Is neither "bold" (bet exactly what is needed to reach the goal) nor "timid" (bet a fixed amount each time)

The CPDO goal was to double the initial stake

Dubins & Savage: There is at most a 50% probability of doubling your stake in a casino, regardless of strategy

A real-world CPDO was a billion-dollar gamble with, at most, a 50% chance of achieving the goal

## More trouble

If the coin is two-tailed then CPDO will Cash-Out in 10 tosses

With a fair coin, CPDO is still doomed by an early string of tails

Sadly, ...

Theorem: Even with a two-headed coin, it is impossible for CPDO to double the stake in a finite lifetime

CPDO's stake then increases on each toss by amounts that form a decreasing geometric series

This series cannot sum to 1 in finitely many tosses

## Zeno's paradox of motion, "The Dichotomy"

---

Atalanta, a fast runner, runs 1 mile in minute 1, 1/2 mile in minute 2, 1/4 mile in minute 3, etc.

Atalanta's incremental distances form a geometric series whose sum is less than 2 for any finite time

The connection with Zeno's paradox was missed by CPDO inventors, buyers, rating agencies, central banks, regulators

**Moulinath Banerjee** (Univ. Michigan)

Our lame excuse: Hein Hundal and I aren't quants or derivatives analysts

Compared to regulators, central banks, etc., we have little resources to study the subject

## Probabilistic analysis of CPDOs

CPDO starts with stake of 1 unit: =  $\underbrace{0.1}_{\text{capital } (\gamma)} + \underbrace{0.9}_{\text{bank loan } (1-\gamma)}$

Stake: 1 unit (\$100 million in the real world)

Goal: To amass a stake of 2 units, then repay bank

Strategy: On each coin toss, bet 1% of: 2 minus current stake

Cash-In Event: The stake reaches 2 units

Cash-Out Event: The stake reaches 0.9 unit or less

Cash-Out  $\longleftrightarrow$  Margin call from bank

$C_1, C_2, C_3, \dots$ : The outcomes of successive coin tosses

$$C_k = \begin{cases} +1, & \text{if the } k \text{ th toss results in heads} \\ -1, & \text{if the } k \text{ th toss results in tails} \end{cases}$$

$Y_k$ : CPDO's stake at the  $k$ th toss,  $k = 0, 1, 2, \dots$

Initial stake:  $Y_0 = 1$

$\delta = 0.01$ : The proportion of  $2 - Y_{k-1}$  that is bet at the  $k$ th toss

$$Y_k = Y_{k-1} + \delta(2 - Y_{k-1})C_k, \quad k \geq 1$$

Net profit (or **loss**) at the  $k$  th stage:  $X_k = Y_k - 1$

$$X_0 = 0; \quad X_k = X_{k-1} + \delta(1 - X_{k-1})C_k, \quad k \geq 1$$

Cash-In Event:  $X_k = 1$

Cash-Out Event:  $X_k \leq -\gamma$

Solve the recurrence relation to get an explicit formula:

$$X_k = 1 - (1 - \delta C_1)(1 - \delta C_2) \cdots (1 - \delta C_k)$$

Two-headed coin ( $C_k \equiv 1$ ):

$$X_k = 1 - (1 - \delta)^k < 1 \quad \text{for all } k$$

We can never reach a profit of 1 with finite  $k$  (Atalanta)

Two-tailed coin ( $C_k \equiv -1$ ):  $X_k = 1 - (1 + \delta)^k$

Solve:  $1 - (1 + \delta)^k \leq -\gamma$

$$k = \frac{\ln(1 + \gamma)}{\ln(1 + \delta)} = 9.58$$

Cash-Out in 10 tosses

## The case of the patient banker

Assume (for now) that the coin is fair

$C_1, C_2, C_3, \dots$  are mutually independent random variables

$$P(C_k = 1) = P(C_k = -1) = 1/2$$

$C_k$  has a Bernoulli distribution with probability of success  $1/2$

$X_k$  is a martingale:  $E(X_k | X_1, \dots, X_{k-1}) = X_{k-1}$ , a.s.

$$E(X_k) = 0, \quad V(X_k) = (1 + \delta^2)^k - 1 \quad (\text{exponential increase})$$

Martingale convergence theorems

For large  $k$ , about 50% of  $C_j$  are  $+1$  and about 50% are  $-1$

$$X_k \simeq 1 - (1 - \delta)^{k/2}(1 + \delta)^{k/2} = 1 - (1 - \delta^2)^{k/2} \rightarrow 1$$

How long do we have to wait for  $P(X_k \geq 0.9) = 95\%$  ?

Answer:  $k \simeq 108,218$ , or over 54,000 years

Good news:  $P(\text{Cash-Out})$  eventually goes to 0:

$$\lim_{k \rightarrow \infty} P(X_k < -\gamma) = 0$$

Bad news:  $P(\text{Successive loss})$  eventually goes to 1:

$$\lim_{k \rightarrow \infty} P(X_{k+1} < 0 \mid X_k < 0) = 1$$

Ugly news:  $P(\text{Cash-Out after a loss})$  eventually goes to 1:

$$\lim_{k \rightarrow \infty} P(X_{k+1} \leq -\gamma \mid X_k < 0) = 1$$

The good news hides bad news: For large  $k$ ,

$$P(\text{Cash-Out}) \simeq \Phi\left(\frac{-\ln(1 + \gamma) + k\mu}{\sqrt{k\sigma^2}}\right)$$

where  $\mu = \frac{1}{2} \ln(1 - \delta^2)$ ,  $\sigma^2 = \frac{1}{4} \left(\ln \frac{1+\delta}{1-\delta}\right)^2$

$P(\text{Cash-Out})$  increases from 2.67% (at  $k = 25$ ) to 33.12% (at  $k = 1,906$ ), and then decreases to 0

Even if CPDO survives the early coin tosses,  $P(\text{Cash-Out})$  increases for a long time before it decreases

## Probability of a Net Loss on the $k$ th Toss

---

$k$	$P(X_k < 0)$	$P(X_k < 0   X_1 < 0)$
1	50%	100%
2	25%	50%
3	50%	75%
4	31%	50%
5	50%	68%
6	34%	50%
7	50%	65%
8	36%	50%
9	50%	63%
10	38%	50%

---

## Undergraduate-level calculations

Theorem:  $P(X_k < 0) \rightarrow 0$  as  $k \rightarrow \infty$ . However,

$$P(X_{2m} < 0) = \frac{1}{2} - \frac{1}{2^{2m+1}} \binom{2m}{m}, \quad m = 1, 2, \dots, 199$$

$$P(X_{2m+1} < 0) = \frac{1}{2}, \quad m = 0, 1, 2, \dots, 99$$

Moreover, CPDO's troubles begin with the very first toss

Theorem:  $P(X_k < 0 | X_1 < 0) \rightarrow 0$  as  $k \rightarrow \infty$ . However,

$$P(X_{2m} < 0 | X_1 < 0) = \frac{1}{2}, \quad m = 1, 2, \dots, 199$$

$$P(X_{2m+1} < 0 | X_1 < 0) = \frac{1}{2} + \frac{1}{2^{2m+1}} \binom{2m}{m}, \quad m = 0, 1, \dots, 198$$

Similar results hold for  $P(X_k < 0 | X_2 < 0)$ , etc.

Theorem:  $P(\text{Successive losses}) \simeq 1$  for a long time:

$$P(X_{2m+1} < 0 | X_{2m} < 0) = 1, \quad m = 1, 2, \dots, 99$$

$$P(X_{2m+2} < 0 | X_{2m+1} < 0) = 1 - \frac{1}{2^{2m+1}} \binom{2m+1}{m+1}, \quad m = 0, \dots, 98$$

Even worse, a net loss at the  $j$ th toss increases the probability of Cash-Out on every future toss

Theorem: For all  $j$  and  $k$ ,

$$P(X_{j+k} \leq -\gamma | X_j < 0) \geq P(X_k \leq -\gamma)$$

## The case of the impatient banker

Futile goal: Cash-In in a finite lifetime

Modified Cash-In Event:  $X_k \geq 1 - \gamma$

Theorem: For a CPDO player subject to margin calls,

With probability 1, the game will end in finite time and

$$P(\text{Cash-Out}) > \gamma + \delta + \gamma\delta = 88.9\%$$

$$P(\text{Modified Cash-In}) < 11.1\%$$

Simulate 1,000 CPDOs each betting 50,000 times

They either are bankrupted by margin calls (nearly 89%)

Or they survive and reach high net profit (nearly 11%)

So, what happened to the real-world CPDOs?

Our analysis overlooked:

Costs of tossing the coin (commissions, trader compensation)

Capital gains taxes

Interest on bank loans

Front-running! (As CPDO is about to reset into the index, other traders can anticipate the spread changes and trade similarly)

If CPDO needs to buy, then others will buy ahead of CPDO, run up the price, and sell out to CPDO, etc.

CPDOs were first sold in 2006 when the economy was white-hot

As the economy slowed, CPDO was stuck with a two-tailed coin

To recover from losses, CPDOs probably accelerated “dynamic hedging,” tossing the coin even faster

This gave them more tails

Dynamic hedging can increase the probability of subsequent tails by depressing credit indices

If the coin is slightly unfair, the probabilities of Cash Out increase dramatically

Real-world CPDOs carried more leverage than in our analysis

Notional amounts up to 15X capital, and then CPDO has  $\gamma = 0.0625$  capital and  $1 - \gamma = 0.9375$  bank debt

A margin limit of 0.625 units leaves little room for error

CPDOs probably started with a bet proportion of  $\delta = 1\%$

They probably increased  $\delta$  to try to recoup early losses

This simply accelerated their losses, and down they went

We suspect that Moody's problems were due, not to software errors, but to the flawed nature of CPDOs

## What were CPDO's credit-default swaps?

---

Insurance policies covering losses on mortgages or bonds

Bond traders bought CDS's (from CPDO) to limit their losses

CPDO was an under-capitalized, unregulated insurance co.

It wrote policies against foreclosures when subprime mortgages, liar loans, mortgage fraud, etc., were at their hottest ever

CPDOs also sold CDS indices, i.e.,

CPDOs insured entire, adjacent neighborhoods in many cities just as home buyers lit financial fireworks in their living rooms

A fire in one house ignited fires in adjacent houses, then through the neighborhood, and eventually across the city

## References

---

Requiem for the CPDO: No flowers

[efinancialnews.com](http://efinancialnews.com)

International Monetary Fund, pp. 51–52

European Central Bank, p. 5, 22–23 (tacit approval), 85–86

Bank of England, p. 199

Bank for International Settlements, p. 39, 60–72

D.R. and Hein Hundal, [CPDO paper](#)