MIMO Capacities : Eigenvalue Computation through Representation Theory

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Outline

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What is MIMO?

- MIMO : Multiple-Input-Multiple-Output.
- Multiple antennas used to transmit and receive signals in wireless communications.
- Use of multiple antennas increase information throughput substantially.

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Asymptotic ergodic capacity of the channels

- Mutual information I averaged over channel realizations for large number of antennas.
- We are interested in the moments of the capacity and the probability of outage.
- Here we discuss a method for computing the moment generating function of I.
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System model

- Signals are complex realizations of the form $re^{i\theta}$.
- n_t : # transmitter antennas.
  n_r : # receiver antennas.
  $\mathbf{x}$: transmitted signals.
  $\mathbf{y}$: received signal.
- $$\mathbf{y} = \mathbf{G}\mathbf{x} + \mathbf{z}$$
- $\mathbf{G}$: Complex $n_r \times n_t$ matrix of Channel coefficients.
  i.e. G_{ij} = channel coefficient : transmitter j to receiver i.
- $\mathbf{z}$: additive noise.

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Model assumptions

- $\mathbf{x}$, $\mathbf{z}$ independent. Their elements are i.i.d.
  We assume mean 0, variance 1, Gaussian structure.
  Arbitrary covariances : easy extensions.
- $\mathbf{G}$ is known to receiver, not transmitter.
- $\mathbf{G}$ has complex Gaussian entries.
  Covariance assumptions later.

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Mutual information

- Capacity

$$I(\mathbf{y};\mathbf{x}|\mathbf{G}) = \log\det(I + \mathbf{G}^\dagger \mathbf{G})$$

- I expressed in nats.
- Moment generating function :

$$g(z) = E[e^{zI}] = E_{\mathbf{G}}[\{\det(I + \mathbf{G}^\dagger \mathbf{G})\}^z]$$

- Probability of outage :

$$P_{out} = E_{\mathbf{G}}[\Theta(I - I_{out})] = \int \frac{g(iz)}{2\pi i} \frac{e^{-izI_{out}}}{z - i0^+} dz$$

where Θ is the Heaviside function.

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Eigenvalues of $\mathbf{G}^\dagger\mathbf{G}$

- Clearly,

$$g(z) = E\left[\prod_{i=1}^{N}(1+\lambda_i)^z\right] = \int_0^\infty \cdots \int_0^\infty \prod (1+\lambda_i)^z p(\lambda) d\lambda$$

where $\lambda = (\lambda_1, \ldots, \lambda_N)$ are the positive eigenvalues of $\mathbf{G}^\dagger\mathbf{G}$.

- $N = \min(n_t, n_r)$. Also, let $M = \max(n_t, n_r)$
- Simple case : $\mathbf{G}$ i.i.d. with zero mean, i.e.

$$p(\mathbf{G}) \propto e^{-tr(\mathbf{G}^\dagger\mathbf{G})}$$

then,

$$p(\lambda) = C_{M,N}\Delta(\lambda)^2 \prod [\lambda_i^{M-N} e^{-\lambda_i}]$$

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Channels with non-trivial distributions

- Semi-correlated channels : non-zero correlation in the transmitters (alternately receivers).

$$p(\mathbf{G}) = c(\mathbf{T})e^{-tr\mathbf{T}^{-1}\mathbf{G}^{\dagger}\mathbf{G}}$$

  for some $\mathbf{T} > 0$.

- Non-zero means : $\mathbf{G}$ has mean $\mathbf{G}_0$

$$p(\mathbf{G}) \propto e^{-tr\left\{(\mathbf{G}-\mathbf{G}_0)^{\dagger}(\mathbf{G}-\mathbf{G}_0)\right\}}$$

- Fully correlated channels : non-zero correlation in the transmitters and the receivers.

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Singular Value Decomposition of G

- The singular values of **G**

$$\mu_i = \sqrt{\lambda_i}$$

Let $\Omega = diag(\mu_1, \ldots, \mu_N)$

- **U**, **V** are unitary matrices,

$$\mathbf{G} = \mathbf{U}\Omega\mathbf{V}^\dagger$$

- Using normalized Haar measures,

$$p(\lambda) = C_{M,N}\Delta(\lambda)^2 \prod_{i=1}^{N} \lambda_i^{M-N} \int\int p(\mathbf{U}\Omega\mathbf{V}^\dagger) d\mathbf{U} d\mathbf{V},$$

integrating over the unitary groups of order n_t and n_r.

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Representation theory: recap

- $\rho : G \to V$, a homomorphism from a group G to a group of invertible matrices V.
- Example : $GL(M)$- complex $M \times M$ invertible matrices, $U(M)$- its subgroup of unitary matrices.
- ρ is irreducible if it has no non-trivial decomposition.
- The irreducible polynomial representations of $U(M)$ are parametrized by $\mathbf{m} = (m_1, \ldots, m_M)$ with integers $m_1 \geq \ldots \geq m_N \geq 0$.
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Dimension and orthogonality

- Dimension of an irreducible representation : dimension of its invariant subspace. For $U(M)$

$$d_{\mathbf{m}} = [\prod_{i=1}^{M} \frac{1}{(M-i)!}](-1)^{M(M-1)/2}\Delta(\mathbf{k})$$

where $k_i = m_i - i + M$.

- For irreducible representations $\mathbf{U}^{(\mathbf{m})}$ and $\mathbf{U}^{(\mathbf{m}')}$,

$$\int (U^{(\mathbf{m})})_{ij}(U^{(\mathbf{m}')\dagger})_{kl} d\mathbf{U} = \frac{\delta_{\mathbf{m}\mathbf{m}'}\delta_{ik}\delta_{jl}}{d_{\mathbf{m}}}$$

an orthogonality property we use later.

- Remember that we need to evaluate

$$\int\int p(\mathbf{U}\Omega\mathbf{V}^{\dagger})d\mathbf{U}d\mathbf{V}$$

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Character of representation

- Character : trace of the representation, $\chi(g) = tr(\rho(g))$.
- $\mathbf{A}^{(\mathbf{m})}$ is the $\mathbf{m}$ representation of $\mathbf{A}$, a $d_{\mathbf{m}}$ dimensional matrix.
- Character of irreducible representations :

$$\chi_{\mathbf{m}}(\mathbf{A}) = tr[\mathbf{A}^{(\mathbf{m})}] = \frac{\det\left(a_i^{m_j+M-j}\right)}{\Delta(a_1, \ldots, a_m)}$$

where a_i are the eigenvalues of $\mathbf{A}$.

- Character expansion of exponential :

$$exp(t\, tr(\mathbf{A})) = \sum_{\mathbf{m}} \alpha_{\mathbf{m}}(t)\chi_{\mathbf{m}}(\mathbf{A})$$

- $\alpha_{\mathbf{m}}(t)$- coefficient of each character in the expansion.

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Semi-correlated channels

- Recall that, $p(\mathbf{G}) = c(\mathbf{T})etr(-\mathbf{T}^{-1}\mathbf{G}^{\dagger}\mathbf{G})$.
- Define $\Lambda = diag(\lambda) = \Omega^2$. Therefore,

$$\begin{aligned}
&\int\int p(\mathbf{U}\Omega\mathbf{V}^{\dagger})d\mathbf{U}d\mathbf{V} \\
&= \int etr(-\Lambda\mathbf{U}^{\dagger}\mathbf{T}^{-1}\mathbf{U})d\mathbf{U} \\
&= \int\sum_{\mathbf{m}}\alpha_{\mathbf{m}}(-1)\chi_{\mathbf{m}}(\Lambda\mathbf{U}^{\dagger}\mathbf{T}^{-1}\mathbf{U})d\mathbf{U} \\
&= \int\sum_{\mathbf{m}}\alpha_{\mathbf{m}}(-1)tr(\Lambda^{(\mathbf{m})}\mathbf{U}^{\mathbf{m}\dagger}(T^{(\mathbf{m})})^{-1}\mathbf{U}^{(\mathbf{m})})d\mathbf{U} \\
&= \sum_{\mathbf{m}}\frac{\alpha_{\mathbf{m}}(-1)}{d_{\mathbf{m}}}\chi_{\mathbf{m}}(\mathbf{T}^{-1})\chi_{\mathbf{m}}(\Lambda)
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Cauchy-Binet summation

- Let $\Upsilon = diag(\tau)$, the eigenvalues of $\mathbf{T}^{-1}$.
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$$\sum_{\mathbf{m}} \Big[\prod_{i=1}^{n_t} \frac{(-1)^{m_i}}{(m_i - i + n_t)!}\Big] \frac{\det(\tau_i^{m_j-j+n_t})\det(\lambda_i^{m_j-j+n_t})}{\Delta(\tau)\Delta(\lambda)}$$

$$\propto \sum_{k_1>\ldots k_{n_t}\geq 0} \Big[\prod_{i=1}^{n_t}\frac{(-1)^{k_i}}{k_i!}\Big]\frac{\det(\tau_i^{k_j})\det(\lambda_i^{k_j})}{\Delta(\tau)\Delta(\lambda)}, (k_i = m_i - i + n_t)$$

$$= \Big[\prod_{i=1}^{n_t}\tau_i^{n_r}\Big]\frac{\det(e^{-\tau_i\lambda_j})}{\Delta(\tau)\Delta(\lambda)}$$

- Recall the Cauchy-Binet formula :

$$\sum_{k_1>\ldots k_{n_t}\geq 0}\det(a_i^{k_j})\det(b_i^{k_j})\prod w(k_i) = \det(W(a_i b_j))$$

where $W(z) = \sum_{i=0}^{\infty} w(i)z^i$.

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Continuation

- Computation of m.g.f. :

$$g(z) \propto \int \cdots \int \prod [(1+\lambda_i)^z \lambda_i^{M-N}] \frac{\Delta(\lambda)}{\Delta(\tau)} \prod \tau_i^{n_r} \det(e^{-\tau_i \lambda_j})$$
$$= (-1)^{n_t(n_t-1)/2} \prod_{j=1}^{n_t} \tau_j^{n_r} \frac{\det L_z}{\Delta(\tau)}$$

  where $L_{z,ij}$ is a confluent hypergeometric function of τ_i's.

- The case of non-zero mean, uncorrelated channels follows similarly.
- Difficulty arises with fully correlated channels.

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Real matrices and orthogonal groups

- Remember we had the nice result

$$\int etr(-\Lambda \mathbf{U}^{\dagger}\mathbf{T}^{-1}\mathbf{U})d\mathbf{U} = \sum_{\mathbf{m}} \frac{\alpha_{\mathbf{m}}(-1)}{d_{\mathbf{m}}} \chi_{\mathbf{m}}(\mathbf{T}^{-1})\chi_{\mathbf{m}}(\Lambda)$$

- Do we have analogous results for the orthogonal matrices?