Log-Convexity Properties of Schur Functions and Hypergeometric Functions of Matrix Argument

Donald Richards

Penn State University

$\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ in $\mathbb{R}^n$

$\lambda \vee \mu := (\max(\lambda_1, \mu_1), \ldots, \max(\lambda_n, \mu_n))$

$\lambda \wedge \mu := (\min(\lambda_1, \mu_1), \ldots, \min(\lambda_n, \mu_n))$

These operations induce an ordering on $\mathbb{R}^n$

Weyl chamber: $\mathcal{W} = \{\lambda \in \mathbb{R}^n : \lambda_1 \geq \cdots \geq \lambda_n\}$

For $\lambda \in \mathcal{W}$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the Schur functions:

$$s_\lambda(x) = \frac{\det({x_i}^{\lambda_j + n - j})}{\prod_{i<j}(x_i - x_j)} \equiv \frac{\Delta_\lambda(x)}{\Delta_0(x)}$$

Algebra (symmetric functions)

Combinatorics (Young tableaux)

Mathematical statistics (complex zonal polynomials)

Representation theory of GL$(n, \mathbb{C})$ (characters of irreducible representations)

Harmonic analysis (spherical functions)

x is a $n \times n$ Hermitian matrix with eigenvalues $x_1, \ldots, x_n$

We shall prove:

Theorem 1: For $\lambda, \mu \in \mathcal{W}$ and $x \in \mathbb{R}_+^n$,

$$s_{\lambda\vee\mu}(x)\, s_{\lambda\wedge\mu}(x) - s_\lambda(x) s_\mu(x) \geq 0\,;$$

i.e., the map $\lambda \mapsto s_\lambda(x)$ is "MTP$_2$"

Theorem 2 (Lam, *et al.*, 2007): For partitions λ, μ, $s_{\lambda\vee\mu}\, s_{\lambda\wedge\mu} - s_\lambda s_\mu$ is monomial-positive and Schur-positive

Theorem 1 applies to all $\lambda, \mu \in \mathcal{W}$, but proves positivity only

Theorem 2 applies to partitions λ, μ only, but proves positivity of all coefficients in the monomial or Schur expansions

Theorem 2 implies Theorem 1 if λ, μ are partitions

MTP$_2$: Mathematical statistics

Log-supermodular: Combinatorics, game theory, economics

FKG: Probability, mathematical physics

Differing names for the same concept

Lorentz; Battle-Rosen; Karlin-Rinott; Herbst-Pitt: An analytical characterization of positive MTP$_2$ functions

A C^2 function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ is MTP$_2$ iff

$$\frac{\partial^2}{\partial\lambda_i\partial\lambda_j}\log\phi(\lambda) \geq 0 \quad \text{for all } i \neq j$$

To study the sign of $\partial^2 \log s_\lambda / \partial\lambda_i \partial\lambda_j$, it suffices to take $(i,j) = (1,2)$

Recall the notation: $\Delta_\lambda(x) = \det(x_i^{\lambda_j+n-j})$

$$\Delta_\lambda^2 \cdot \frac{\partial^2}{\partial\lambda_1\lambda_2} \log s_\lambda = \Delta_\lambda \cdot \frac{\partial^2}{\partial\lambda_1\lambda_2}\Delta_\lambda - \frac{\partial}{\partial\lambda_1}\Delta_\lambda \cdot \frac{\partial}{\partial\lambda_2}\Delta_\lambda$$

D.R., “Total positivity properties of generalized hypergeometric functions of matrix argument,” J. Statist. Phys., 2004

Positivity properties of determinants of classical generalized hypergeometric functions

Karlin (1968): For vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_{n-2} \in \mathbb{R}^n$, define

$$D(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_{n-2}) = \det(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_{n-2})$$

Then, for $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{b}_1, \boldsymbol{b}_2 \in \mathbb{R}^n$,

$$\begin{vmatrix} D(\boldsymbol{a}_1, \boldsymbol{b}_1, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_{n-2}) & D(\boldsymbol{a}_1, \boldsymbol{b}_2, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_{n-2}) \\ D(\boldsymbol{a}_2, \boldsymbol{b}_1, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_{n-2}) & D(\boldsymbol{a}_2, \boldsymbol{b}_2, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_{n-2}) \end{vmatrix}$$

$$= D(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_{n-2}) \cdot D(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_{n-2})$$

Sylvester's formula for compound determinants

Let $K(x_i, \lambda_j) \equiv x_i^{\lambda_j + n - j}$ and set

$$\boldsymbol{a}_1 = \begin{pmatrix} K(x_1, \lambda_1) \\ \vdots \\ K(x_n, \lambda_1) \end{pmatrix}, \quad \boldsymbol{a}_2 = \frac{\partial}{\partial \lambda_1} \begin{pmatrix} K(x_1, \lambda_1) \\ \vdots \\ K(x_n, \lambda_1) \end{pmatrix},$$

$$\boldsymbol{b}_1 = \begin{pmatrix} K(x_1, \lambda_2) \\ \vdots \\ K(x_n, \lambda_2) \end{pmatrix}, \quad \boldsymbol{b}_2 = \frac{\partial}{\partial \lambda_2} \begin{pmatrix} K(x_1, \lambda_2) \\ \vdots \\ K(x_n, \lambda_2) \end{pmatrix},$$

$$\boldsymbol{f}_j = \begin{pmatrix} K(x_1, \lambda_j) \\ \vdots \\ K(x_n, \lambda_j) \end{pmatrix}, \quad j = 1, \ldots, n-2$$

$$D(\boldsymbol{a}_1, \boldsymbol{b}_1, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_{n-2}) = \Delta_\lambda(x)$$

$$D(\boldsymbol{a}_1, \boldsymbol{b}_2, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_{n-2}) = \frac{\partial}{\partial \lambda_2} \Delta_\lambda(x)$$

$$D(\boldsymbol{a}_2, \boldsymbol{b}_1, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_{n-2}) = \frac{\partial}{\partial \lambda_1} \Delta_\lambda(x)$$

$$D(\boldsymbol{a}_2, \boldsymbol{b}_2, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_{n-2}) = \frac{\partial^2}{\partial \lambda_1 \partial \lambda_2} \Delta_\lambda(x)$$

By Karlin's identity,

$$
\begin{aligned}
\Delta_\lambda^2 \cdot \frac{\partial^2}{\partial\lambda_1\lambda_2} \log s_\lambda \\
&= \Delta_\lambda \cdot \frac{\partial^2}{\partial\lambda_1\lambda_2}\Delta_\lambda - \frac{\partial}{\partial\lambda_1}\Delta_\lambda \cdot \frac{\partial}{\partial\lambda_2}\Delta_\lambda \\
&= D(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{f}_1, \dots, \boldsymbol{f}_{n-2}) \cdot D(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{f}_1, \dots, \boldsymbol{f}_{n-2})
\end{aligned}
$$

$$D(\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{f}_1,\ldots,\boldsymbol{f}_{n-2})$$

$$\equiv \begin{vmatrix} K(x_1,\lambda_1) & \frac{\partial}{\partial\lambda_1}K(x_1,\lambda_1) & K(x_1,\lambda_3) & \cdots & K(x_1,\lambda_n) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ K(x_n,\lambda_1) & \frac{\partial}{\partial\lambda_1}K(x_n,\lambda_1) & K(x_n,\lambda_3) & \cdots & K(x_n,\lambda_n) \end{vmatrix}$$

$$= \lim_{\lambda_2\to\lambda_1} \frac{\det\bigl(K(x_i,\lambda_j)\bigr)}{\lambda_2-\lambda_1}$$

$K(x,\lambda)$ is STP$_\infty$: $\det(K(x_i,\lambda_j)) > 0$ for $x_1 > \cdots > x_n > 0$, $\lambda_1 > \cdots > \lambda_n$, and all $n \geq 1$

Therefore, the limit is ≤ 0 (numerator > 0; denominator < 0)

Conclusion: For $\lambda \in \mathcal{W}$ and $x \in \mathbb{R}^n_+$,

$$\frac{\partial^2}{\partial\lambda_i\partial\lambda_j} \log s_\lambda(x) > 0, \quad i \neq j$$

Therefore, for $\lambda, \mu \in \mathcal{W}$ and $x \in \mathbb{R}^n_+$,

$$s_{\lambda\vee\mu}(x)\, s_{\lambda\wedge\mu}(x) - s_\lambda(x)s_\mu(x) \geq 0$$

Possible extensions and generalizations

Positivity properties of $\dfrac{\partial^2}{\partial\lambda_1 \partial x_1} \log s_\lambda(x)$; see

D.R., J. Statist. Phys., 2004: Dodgson’s condensation formula

Apply more general forms of Sylvester’s formula

MTP$_k$, $k > 2$

Non-commutative Sylvester's determinantal formula

Monomial/Schur-positivity of q-Schur difference products

Characters of other compact Lie groups: Start with Weyl's formula and carry out derivatives

$\mathrm{SO}(5)$: See Gross and D.R., J. Approx. Theory, 1995

Fortuin-Kasteleyn-Ginibre, J. Math. Phys., 1971

ϕ: MTP$_2$ probability density function (p.d.f.) on $\mathbb{R}^n$

FKG Inequality: If f, g are increasing functions on $\mathbb{R}^n$ then $E(fg) - E(f)E(g) \geq 0$

D.R., Ann. Probab., 2004

Generalized FKG Inequalities: If f, g, h are nonnegative increasing functions on $\mathbb{R}^n$ then

$$\begin{aligned} 2E(fgh) - [E(fg)E(h) + E(fh)E(g) + E(f)E(gh)] \\ + E(f)E(g)E(h) \geq 0 \end{aligned}$$

$\mathcal{P}_n$: The set of partitions of length $\leq n$

x, y: $n \times n$ Hermitian positive definite matrices

c_λ: Coefficients that are MTP$_2$ in λ

Construct a MTP$_2$ p.d.f. on $\mathcal{P}_n$: $\quad \phi(\lambda) = \dfrac{c_\lambda s_\lambda(x)}{\sum_{\mu \in \mathcal{P}_n} c_\mu s_\mu(x)}$

FKG: If f and g are increasing then

$$\Big(\sum_{\lambda \in \mathcal{P}_n} c_\lambda s_\lambda(x) f_\lambda g_\lambda\Big) \cdot \Big(\sum_{\lambda \in \mathcal{P}_n} c_\lambda s_\lambda(x)\Big) \geq \Big(\sum_{\lambda \in \mathcal{P}_n} c_\lambda s_\lambda(x) f_\lambda\Big) \cdot \Big(\sum_{\lambda \in \mathcal{P}_n} c_\lambda s_\lambda(x) g_\lambda\Big)$$

Partitional shifted factorial: $(a)_\lambda = \prod_{j=1}^n (a - j + 1)_{\lambda_j}$

Observe that $(a)_{\lambda \vee \mu} \, (a)_{\lambda \wedge \mu} \equiv (a)_\lambda (a)_\mu$

For $a_1, \ldots, a_p, b_1, \ldots, b_q > n - 1$ and $y \in \mathbb{R}_+^n$, set

$$c_\lambda = \frac{(a_1)_\lambda \cdots (a_p)_\lambda}{(b_1)_\lambda \cdots (b_p)_\lambda (n)_\lambda} s_\lambda(y)$$

c_λ is a product of MTP$_2$ functions, therefore c_λ is MTP$_2$

Why choose the c_λ as above?

Gross and D.R., J. Approx. Theory, 1989

Relationship between Schur functions and zonal polynomials

Hypergeometric functions of two Hermitian matrix arguments

With the above choice for c_λ, we have

$$\sum_{\lambda \in \mathcal{P}_n} c_\lambda s_\lambda(x) = {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; x, y\right)$$

For $a_{p+1}, b_{q+1} > n-1$, set $f(\lambda) = (a_{p+1})_\lambda$ and $g(\lambda) = 1/(b_{q+1})_\lambda$

f is increasing and g is decreasing on $\mathcal{P}_n$, so we apply FKG

Theorem: Let x, y be $n \times n$ Hermitian p.d. matrices, $p \leq q$, and $a_1, \ldots, a_{p+1}, b_1, \ldots, b_{q+1} > n-1$. Then,

$$
{}_{p+1}F_{q+1}\binom{a_1, \ldots, a_{p+1}}{b_1, \ldots, b_{q+1}}; x, y\Big) \; {}_pF_q\binom{a_1, \ldots, a_p}{b_1, \ldots, b_q}; x, y\Big)
$$

$$
\leq \; {}_{p+1}F_q\binom{a_1, \ldots, a_{p+1}}{b_1, \ldots, b_q}; x, y\Big) \; {}_pF_{q+1}\binom{a_1, \ldots, a_p}{b_1, \ldots, b_{q+1}}; x, y\Big)
$$

Apply the generalized FKG inequalties

Obtain more general sums-of-products inequalities for the generalized hypergeometric ${}_pF_q(x, y)$ functions