Zonal Polynomials and Hypergeometric Functions of Matrix Argument

Some matrix spaces

$G = GL(n, \mathbb{R})$: The general linear group, containing all $n \times n$ nonsingular real matrices

$S = S(n, \mathbb{R})$: The space of real symmetric matrices

G “acts” on S: $x \in G$ acts on $s \in S$ by $x \circ s \equiv xsx'$

$$\begin{aligned} x_1x_2 \circ s &= (x_1x_2)s(x_1x_2)' = x_1x_2sx_2'x_1' \\ &= x_1(x_2sx_2')x_1' = x_1(x_2 \circ s)x_1' = x_1 \circ (x_2 \circ s) \end{aligned}$$

$x_1x_2 \circ s = x_1 \circ (x_2 \circ s)$

This is a group action

right-action vs. left-action

Notation: For $x = (x_{ij}) \in G$,

$$\Delta(x) := \det(x)$$

$$\Delta_j(x) = \Delta \begin{pmatrix} x_{11} & \cdots & x_{1j} \\ \vdots & & \vdots \\ x_{j1} & \cdots & x_{jj} \end{pmatrix}, \qquad j = 1, \ldots, n$$

The *standard bitriangular structure of* G: Each $x \in G$ can be expressed as $x = vcu$ where

c is diagonal

u is upper triangular with 1's on the main diagonal

v is lower triangular with 1's on the main diagonal

The map from $(u, c, v) \to vcu$ is smooth when restricted to the subset of matrices $x \in G$ such that

$$\prod_{j=1}^{n} \Delta_j(x) \neq 0$$

$O(n)$: the group of $n \times n$ orthogonal matrices

$O(n)$ is a *maximal compact subgroup* of G

$P(n, \mathbb{R})$: The cone of positive definite $n \times n$ matrices

Each $r \in P(n, \mathbb{R})$ is of the form $r = xx'$, $x \in G$

Polynomials on G

ϕ: A polynomial function on G

$\phi(x)$ is a polynomial in the entries x_{ij} of x

$\mathcal{P}(G)$: The space of polynomials on G

$\mathcal{P}_d(G)$: The space of polynomials homogeneous of degree d

$\mathcal{P}_d(G)$ is a vector space of dimension $\binom{N+d-1}{N-1}$, where $N \equiv n^2$

G is an open subset of $\mathbb{R}^{n\times n}$

ϕ extends uniquely to a polynomial on $\mathbb{R}^{n\times n}$

A polynomial on G restricts uniquely to a polynomial on $P(n,\mathbb{R})$

For $a \in G$, define $R_a : \mathcal{P}(G) \to \mathcal{P}(G)$ by:

$$R_a \phi(x) = \phi(xa), \quad x \in G$$

R_a is called the *right regular representation* of G on $\mathcal{P}(G)$

What is R_a if $a = I_n$?

$R_{ab}\phi = R_b R_a \phi$

$\mathcal{P}_d(G)$ is invariant under R_a: $R_a \mathcal{P}_d(G) = \mathcal{P}_d(G)$

Regard $\mathbb{R}^{n \times n}$ as $\mathbb{R}^N$

Each polynomial ϕ on $\mathbb{R}^{n \times n}$ is a polynomial on $\mathbb{R}^N$

$$\phi(x) = \sum_{\alpha_1,\ldots,\alpha_N} a_{\alpha_1,\ldots,\alpha_N} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N}$$

Shorthand notation:

$$\phi(x) = \sum_{\alpha} a_\alpha X^\alpha$$

$$\alpha! = \alpha_1! \cdots \alpha_N!$$

Conjugate of ϕ: $\tilde{\phi}(x) = \sum_\alpha \bar{a}_\alpha X^\alpha$

The differentiation inner product on $\mathcal{P}(G)$

For $\phi = \sum_\alpha a_\alpha X^\alpha$ and $\psi = \sum_\alpha b_\alpha X^\alpha$, define the inner product

$$\langle \phi | \psi \rangle = \sum_\alpha \alpha! a_\alpha \bar{b}_\alpha$$

This inner product arises from differentiation of ψ by ϕ

HW: Prove that for $k \in O(n)$, R_k is a unitary operator on $\mathcal{P}(G)$:

$$\langle R_k \phi | R_k \psi \rangle = \langle \phi | \psi \rangle$$

Hint: Show that $\langle R_{a'} \phi | R_{a^{-1}} \psi \rangle = \langle \phi | \psi \rangle$, $a \in G$

Under the inner product,

$$\mathcal{P}(G) = \sum_{d=0}^{\infty} \oplus \mathcal{P}_d(G)$$

Representations of G

G: $GL(n,\mathbb{R})$

V: A space of linear transformations

$\rho : G \to V$ is a *representation* of G on V if

$$\rho(xy) = \rho(x)\rho(y), \quad x, y \in G$$

The dimension of ρ is the dimension of V

Example: The right regular representation

Example: $\rho(x) = \Delta(x)^k \overline{\Delta(x)}^l$ is a one-dimensional representation of G

All one-dimensional representations of G are powers of $\Delta(x)$

Given representations ρ_1, ρ_2, form the direct sum

$$(\rho_1 \oplus \rho_2(x) = \begin{pmatrix} \rho_1(x) & 0 \\ 0 & \rho_2(x) \end{pmatrix}$$

This gives us a new representation

Irreducible representations: Those which cannot be written as a direct sum of lower-dimensional representations

The basic problem in representation theory: Describe all irreducible representations of a group

$G = GL(n, \mathbb{R})$: We use the standard bitriangular decomposition to describe some irreducible polynomial representations of G

Each irreducible, finite-dimensional, polynomial representation of $GL(n,\mathbb{R})$ is parametrized by an n-tuple $(m_1,\ldots,m_n)$ of integers where $m_1 \geq \cdots \geq m_n \geq 0$

π_m: The representation corresponding to the *signature* $m = (m_1,\ldots,m_n)$

Return to the (U,C,V) decomposition: $x = vcu$

For each signature m and $c = \text{diag}(c_1,\ldots,c_n)$, let

$$\mu_m(c) = |c_1|^{2m_1}|c_2|^{2m_2}\cdots|c_n|^{2m_n}$$

μ_m is a *character* of C

μ_m is a one-dimensional representation of C

$\mathcal{P}^{2m}(G)$: The collection of all $\phi \in \mathcal{P}(G)$ such that

$$\phi(vcx) = \mu_{2m}(c)\phi(x), \quad v \in V, c \in C, x \in \mathbb{R}^{n\times n}$$

Each ϕ in $\mathcal{P}^{2m}(G)$ is homogeneous:

$$\mathcal{P}^{2m}(G) \subset \mathcal{P}_d(G), \quad d = 2|m|, |m| = m_1 + \cdots + m_n$$

A crucial example of a polynomial in $\mathcal{P}^{2m}(G)$:

$$\phi_{2m}(x) = \Delta(x)^{2m_n} \prod_{j=1}^{n-1} \Delta_j(x)^{2(m_j - m_{j+1})}$$

Calculate $\Delta_j(vcx)$ to see why $\phi_{2m} \in \mathcal{P}^{2m}(G)$

More comments on the representation theory of G

Orthogonally invariant polynomials

$\mathcal{I}(G)$: The space of left-invariant polynomials ϕ on G,

$$\phi(kx) = \phi(x), \quad k \in O(n), x \in G$$

If ϕ is homogeneous then it is of even degree, because $-I_n \in O(n)$

$\mathcal{I}_{2d}(G)$: The class of $\phi \in \mathcal{I}(G)$ which are homogeneous of degree $2d$

$$\mathcal{I}(G) = \sum_{d=0}^{\infty} \oplus \mathcal{I}_{2d}(G)$$

The spherical transform: Given $\phi \in \mathcal{P}(G)$, construct $\phi^{\#} \in \mathcal{I}(G)$:

$$\phi^{\#}(x) = \int_{O(n)} \phi(kx)\, dk$$

Apply the spherical transform to each $\phi \in \mathcal{P}^{2m}(G)$ to get the space $\mathcal{I}^{2m}(G)$

The spherical transform decomposes $\mathcal{I}_{2d}(G)$ into irreducible, orthogonal subspaces

$$\mathcal{I}_{2d}(G) = \sum_{|m|=d} \oplus \mathcal{I}^{2m}(G)$$

$S = S(n, \mathbb{R})$: The space of real symmetric matrices

Each polynomial ϕ on G gives rise to a polynomial q on S:

$$q(xx') = \phi(x)$$

The spherical transform converts each left-invariant polynomial p on G into a biinvariant polynomial q on S

The zonal polynomials

Apply the spherical transform to $\mathcal{I}_{2d}(G)$ and $\mathcal{I}^{2m}(G)$

$$\mathcal{P}_d(S) = \sum_{|m|=d} \oplus \mathcal{P}^m(S)$$

This decomposition is *multiplicity free*: Up to constant multiples, there exists only one nontrivial, biinvariant polynomial in $\mathcal{P}^m(S)$

This unique polynomial is called the *zonal polynomial*

Let $\phi \in \mathcal{P}_d(S)$, $\phi(ksk') = \phi(s)$, $k \in O(n)$, $s \in S$. Then there exist *unique* $\phi_m \in \mathcal{P}^m(S)$ such that

$$\phi = \sum_{|m|=d} \phi_m$$

The $\{\phi_m\}$ are orthogonal w.r.t. the differentiation inner product

Is each subspace $\mathcal{P}^m(S)$ nontrivial? Is there an explicit formula for each zonal polynomial?

Apply the spherical transform to the "crucial example" in $\mathcal{P}^{2m}(G)$

$$q_m(s) = \int_{O(n)} \Delta(ksk')^{m_n} \prod_{j=1}^{n-1} \Delta_j(ksk')^{m_j - m_{j+1}} \, dk$$

$q_m \in \mathcal{P}^m(S)$ and $q_m > 0$ on the cone of positive definite matrices

Conclude: Each $\mathcal{P}^m(S)$ is nontrivial

q_m is the *zonal polynomial of weight* m

In multivariate statistical analysis, we choose the zonal polynomials Z_m to be such that

$$(\text{tr}\, s)^d = \sum_{|m|=d} Z_m(s)$$

$$Z_m(s) = Z_m(I_n) \int_{O(n)} \Delta(ksk')^{m_n} \prod_{j=1}^{n-1} \Delta_j(ksk')^{m_j - m_{j+1}} \, dk$$

This requires that we be able to calculate $Z_m(I_n)$

James (1964), Muirhead (1982), Gross and D.R. (1987), etc.

Theorem: Z_m is an eigenfunction of the Laplace-Beltrami operator and, more generally, all G-invariant differential operators on S

D_s: A G-invariant differential operator

$D_s = D_{x'sx}$, $s \in S$, $x \in G$

If $s = (s_{ij}) \in S$, set

$$\partial_s = \left(\tfrac{1}{2}(1 + \delta_{ij})\frac{\partial}{\partial s_{ij}}\right)$$

Examples of invariant differential operators

$$\text{tr}\,(s\partial_s)^k, \quad k = 1, 2, \ldots; \quad \Delta(s\partial_s)$$

tr $(s\partial_s)^2$ is the Laplace-Beltrami operator

It can be shown that

$$q_m(s) = \Delta(s)^{m_n} \prod_{j=1}^{n-1} \Delta_j(s)^{m_j - m_{j+1}}$$

is an eigenfunction of every invariant differential operator D_s

Since D_s is G-invariant then $D_s = D_{ksk'}$

D.R. (SIAM J. Math. Analysis, 1985): Applications of invariant differential operators ...

$$
\begin{aligned}
D_s Z_m(s) &\propto D_s \int_{O(n)} q_m(ksk')\,dk \\
&= \int_{O(n)} D_s q_m(ksk')\,dk \\
&= \int_{O(n)} D_{ksk'} q_m(ksk')\,dk \\
&\propto \int_{O(n)} q_m(ksk')\,dk \\
&= Z_m(s)
\end{aligned}
$$

Theorem: For $s, t \in S$,

$$\int_{O(n)} Z_m(ksk't)\, dk = \frac{Z_m(s)\, Z_m(t)}{Z_m(I_n)}$$

Denote the LHS by $f(s)$

Check that f is an eigenfunction of every invariant D_s

Therefore $f(s) \propto Z_m(s)$

Evaluate f at the identity matrix

We can also evaluate Laplace transforms of Z_m and q_m

The invariant measure on $P(n,\mathbb{R})$: $d_*s = \Delta(s)^{-(p+1)/2} ds$

For $Re(\alpha) > (n-1)/2$,

$$\int_P e^{-\text{tr}\, r} \Delta(r)^\alpha q_m(r) d_*r = [\alpha]_m \Gamma_n(\alpha)$$

where $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) = \Gamma(\alpha+k)/\Gamma(\alpha)$

$$[\alpha]_m = \prod_{j=1}^{n} \left(\alpha - \tfrac{1}{2}(j-1)\right)_{m_j}$$

$$\Gamma_n(\alpha) = \pi^{n(n-1)/4} \prod_{j=1}^{n} \Gamma\left(\alpha - \tfrac{1}{2}(j-1)\right)$$

Proof: Apply the standard bitriangular structure (a.k.a. Bartlett decomposition), etc.

For $z \in P(n, \mathbb{R})$, $s \in S$,

$$\int_P e^{-\text{tr}\, rz} \Delta(r)^\alpha Z_m(rs) d_* r = [\alpha]_m \Gamma_n(\alpha) \Delta(z)^{-\alpha} Z_m(sz^{-1})$$

Denote this integral by $F(s, z)$

We already know $F(I_n, I_n)$

Check that $F(s, I_n)$ is $O(n)$ invariant and is in $\mathcal{P}^m(S)$

Therefore $F(s, I_n) \propto Z_m(s)$

Use a change of variables to show that

$$F(s, z) = \Delta(z)^{-\alpha} F(z^{-1/2} s z^{-1/2}, I_n)$$

For $z \in S$,

$$\int_{0<r<I_n} \Delta(r)^{\alpha} \Delta(I_n - r)^{\beta-\alpha-\frac{1}{2}(n+1)} Z_m(rz)\, d_* r$$

$$= \frac{\Gamma_n(\alpha)\Gamma_n(\beta-\alpha)}{\Gamma_n(\beta)} \frac{[\alpha]_m}{[\beta]_m} Z_m(z)$$

As in the classical setting, apply the convolution formula for the Laplace transform

Hypergeometric functions of matrix argument

Recall:

$$[\alpha]_m = \prod_{j=1}^{n} \left(\alpha - \tfrac{1}{2}(j-1)\right)_{m_j}$$

The generalized hypergeometric function with argument $s \in S$:

$$_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;s) = \sum_{d=0}^{\infty} \sum_{|m|=d} \frac{[\alpha_1]_m \cdots [\alpha_p]_m}{[\beta_1]_m \cdots [\beta_q]_m} \frac{Z_m(s)}{d!}$$

When do these series converge?

Reinhardt domain of convergence: Gross and D.R. (1987), Faraut and Korányi (1994)

Example 1: The series for ${}_0F_0$ converges everywhere on S

$$\begin{aligned}
{}_0F_0(s) &= \sum_{d=0}^{\infty} \sum_{|m|=d} \frac{Z_m(s)}{d!} \\
&= \sum_{d=0} \frac{1}{d!} \sum_{|m|=d} Z_m(s) \\
&= \sum_{d=0} \frac{1}{d!} (\text{tr}\, s)^d \\
&= \exp(\text{tr}\, s)
\end{aligned}$$

Example 2: The series for ${}_1F_0$ converges for $||s|| < 1$

$$\begin{aligned}\Delta(I_n - s)^{-\alpha} &= \frac{1}{\Gamma_n(\alpha)} \int_P e^{-\mathrm{tr}\,(I_n - s)r} \Delta(r)^\alpha \, d_* r \\ &= \frac{1}{\Gamma_n(\alpha)} \int_P e^{-\mathrm{tr}\, r} \, {}_0F_0(rs) \Delta(r)^\alpha \, d_* r \\ &= \frac{1}{\Gamma_n(\alpha)} \sum_{d=0} \frac{1}{d!} \sum_{|m|=d} \int_P e^{-\mathrm{tr}\, r} \Delta(r)^\alpha Z_m(rs) \, d_* r \\ &= \sum_{d=0} \frac{1}{d!} \sum_{|m|=d} \frac{[\alpha]_m Z_m(s)}{d!} \\ &= {}_1F_0(\alpha; s)\end{aligned}$$

Laplace and beta integrals

For $p \leq q$ and $Re(\alpha_{p+1}) > (n-1)/2$,

$$\frac{1}{\Gamma_n(\alpha_{p+1})} \int_P e^{-\text{tr}\, rz} {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; r) \Delta(r)^{\alpha_{p+1}} \, d_* r$$
$$= \Delta(z)^{-\alpha_{p+1}} \; {}_{p+1}F_q(\alpha_1, \ldots, \alpha_p, a_{p+1}; \beta_1, \ldots, \beta_q; z^{-1})$$

Note: If $p = q$ then the RHS converges for $||z^{-1}|| < 1$

$$\int_{0<r<I_n} \Delta(r)^{\alpha_{p+1}} \Delta(I_n - r)^{\beta_{q+1} - \alpha_{p+1} - \frac{1}{2}(n+1)}$$
$$\times \; {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; rs) \, d_* r$$
$$= \frac{\Gamma_n(\alpha_{p+1}) \Gamma_n(\beta_{q+1} - \alpha_{p+1})}{\Gamma_n(\beta_{q+1})}$$
$${}_{p+1}F_{q+1}(\alpha_1, \ldots, \alpha_{p+1}; \beta_1, \ldots, \beta_{q+1}; s)$$

A few references

Herz (1955)

Constantine (1962)

James (1964)

Maass (1971)

Muirhead (1982)

Gross and D.R. (1987)

D.R. (Editor). *Contemp. Math.*, Vol. 138, 1992

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